

STRUCTURE AND  $f$ -DEPENDENCE OF THE A.C.I.M.  
FOR A UNIMODAL MAP  $f$  OF MISIUREWICZ TYPE.

by David Ruelle\*.

**Abstract.** *By using a suitable Banach space on which we let the transfer operator act, we make a detailed study of the ergodic theory of a unimodal map  $f$  of the interval in the Misiurewicz case. We show in particular that the absolutely continuous invariant measure  $\rho$  can be written as the sum of  $1/\text{square root}$  spikes along the critical orbit, plus a continuous background. We conclude by a discussion of the sense in which the map  $f \mapsto \rho$  may be differentiable.*

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## 0 Introduction.

This paper is part of an attempt to understand the smoothness of the map  $f \mapsto \rho$  where  $(M, f)$  is a differentiable dynamical system and  $\rho$  an SRB measure. [For a general introduction to the problems involved, see for instance [2], [31]]. Smoothness has been established for uniformly hyperbolic systems (see [17], [21], [14], [22], [9]). In that case, one finds that the derivative of  $\rho$  with respect to  $f$  can be expressed in terms of the value at  $\omega = 0$  of a *susceptibility function*  $\Psi(e^{i\omega})$  which is holomorphic when the *complex frequency*  $\omega$  satisfies  $\text{Im } \omega > 0$ , and meromorphic for  $\text{Im } \omega > \text{some negative constant}$ . In the absence of uniform hyperbolicity,  $f \mapsto \rho$  need not be continuous. Consider then a family  $(f_\kappa)_{\kappa \in \mathbf{R}}$ . A theorem of H. Whitney [29] gives general conditions under which, if  $\rho_\kappa$  is defined on  $K \subset \mathbf{R}$ , then  $\kappa \mapsto \rho_\kappa$  extends to a differentiable function of  $\kappa$  on  $\mathbf{R}$ . Taking  $\rho_\kappa$  to be an SRB measure for  $f_\kappa$ , this gives a reasonable meaning to the differentiability of  $\kappa \mapsto \rho_\kappa$  on  $K$  (as proposed in [24], see [20], [11] for a different application of Whitney's theorem), even though we start with a noncontinuous function  $\kappa \mapsto \rho_\kappa$  on  $\mathbf{R}$ .

Using Whitney's theorem to study SRB states as proposed above is a delicate matter. A simple situation that one may try to analyze is when  $(M, f)$  is a unimodal map of the interval and  $\rho$  an absolutely continuous invariant measure (a.c.i.m.). [From the vast literature on this subject, let us mention [12], [13], [6], [7], [8], [28]]. A preliminary study of the Markovian case (*i.e.*, when the critical orbit is finite, see [23], [16]) shows that the susceptibility function  $\Psi(\lambda)$  has poles for  $|\lambda| < 0$ , but is holomorphic at  $\lambda = 1$ . This study suggests that in non-Markovian situations  $\Psi$  may have a natural boundary separating  $\lambda = 0$  (around which  $\Psi$  has a natural expansion) and  $\lambda = 1$  (corresponding to  $\omega = 0$ ). Misiurewicz [19] has studied a class of unimodal maps where the critical orbit stays away from the critical point, and he has proved the existence of an a.c.i.m.  $\rho$  for this class. This seems a good situation where one could study the dependence of  $\rho$  on  $f$ , as pointed out to the author by L.-S. Young.

A desirable starting point to study the dependence of the a.c.i.m.  $\rho$  on  $f$  is to have an operator  $\mathcal{L}$  on a Banach space  $\mathcal{A}$  such that  $\mathcal{L}\rho = \rho$ , and 1 is a simple isolated eigenvalue of  $\mathcal{L}$ . The main content of the present paper is the construction of  $\mathcal{A}$  and  $\mathcal{L}$  with the desired properties. Specifically we write  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ , where  $\mathcal{A}_2$  consists of *spikes*, *i.e.*,  $1/\text{square root singularities}$  at points of the critical orbit, which are known to be present in  $\rho$ . We are thus able to prove that the a.c.i.m.  $\rho$  is the sum of a continuous background, and of the spikes (see Theorem 9, and the Remarks 16). Note that the construction of an operator  $\mathcal{L}$  with a spectral gap had been achieved earlier by G. Keller and T. Nowicki [18], and by L.-S. Young [30] (our construction, in a more restricted setting, leads to stronger results).

We start studying the smoothness of the map  $f \mapsto \rho$  by an informal discussion in Section 17. Theorem 19 proves the differentiability along topological conjugacy classes (which are codimension 1) and relates the derivative to the value at  $\lambda = 1$  of a modified susceptibility function  $\Psi(X, \lambda)$ . [Following an idea of Baladi and Smania [5], it is plausible that differentiability in the sense of Whitney holds in directions tangent to a conjugacy class, see below]. Transversally to topological conjugacy classes the map  $f \mapsto \rho$  is continuous, but appears not to be differentiable. While this nondifferentiability is not rigorously proved, it seems to be an unavoidable consequence of the fact that the weight of the  $n$ -th

spike is roughly  $\sim \alpha^{n/2}$  (for some  $\alpha \in (0, 1)$ ) while its speed when  $f$  changes is  $\sim \alpha^{-n}$ . [See Section 16(c). In fact, for a smooth family  $(f_\kappa)$  restricted to values  $\kappa \in K$  such that  $f_\kappa$  is in a suitable Misiurewicz class, the estimates just given for the weight and speed of the spikes suggest that  $\kappa \rightarrow \rho_\kappa(A)$  for smooth  $A$  is  $\frac{1}{2}$ -Hölder, and nothing better, but we have not proved this]. Physically, let us remark that the spikes of high order  $n$  will be drowned in noise, so that discontinuities of the derivative of  $f \mapsto \rho$  will be invisible.

Note that the susceptibility functions  $\Psi(\lambda)$ ,  $\Psi(X, \lambda)$  to be discussed may have singularities both for large  $|\lambda|$  and small  $|\lambda|$ . [The latter singularities do not occur for uniformly hyperbolic systems, but show up for the unimodal maps of the interval in the Markovian case, as we have mentioned above. A computer search of such singularities is of interest [10]].

A study similar to that of the present paper has been made (Baladi [3], Baladi and Smania [5]) for piecewise expanding maps of the interval. In that case it is found that  $f \mapsto \rho$  is not differentiable in general, but Baladi and Smania study the differentiability of  $f \mapsto \rho$  along directions tangent to topological conjugacy classes (horizontal directions), not just for  $f$  restricted to a class. Note that our  $1/\text{square root}$  spikes are replaced in the piecewise expanding case by jump discontinuities. This entails some serious differences, in particular, in the piecewise expanding case  $\Psi(\lambda)$  is holomorphic for  $|\lambda| < 1$ .

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## 1 Setup.

Let  $I$  be a compact interval of  $\mathbf{R}$ , and  $f : \mathbf{R} \rightarrow \mathbf{R}$  be real-analytic. We assume that there is  $c$  in the interior of  $I$  such that  $f'(c) = 0$ ,  $f'(x) > 0$  for  $x < c$ ,  $f'(x) < 0$  for  $x > c$ , and  $f''(c) < 0$ . Replacing  $I$  by a possibly smaller interval, we assume that  $I = [a, b]$  where  $b = fc$ ,  $a = f^2c$ , and  $a < fa$ .

We shall construct a *horseshoe*  $H \subset (a, b)$ , i.e., a mixing compact hyperbolic set with a Markov partition for  $f$ . Following Misiurewicz [19] we shall assume that  $fa \in H$ .

Under natural conditions to be discussed below we shall study the existence of an a.c.i.m.  $\rho(x) dx$  for  $f$ , and its dependence on  $f$ .

## 2 Construction of the set $H(u_1)$ .

Let  $u_1 \in [a, b]$  and define the closed set

$$H(u_1) = \{x \in [a, b] : f^n x \geq u_1 \text{ for all } n \geq 0\}$$

We have thus  $fH(u_1) \subset H(u_1)$ . Assuming that  $H(u_1)$  is nonempty, let  $v$  be its minimum element, then  $H(u_1) = H(v)$ . [Since  $v \in H(u_1)$  we have  $v \geq u_1$ , hence  $H(v) \subset H(u_1)$ . If  $H(u_1)$  contained an element  $w \notin H(v)$  we would have  $H(u_1) \ni f^k w < v$  for some  $k \geq 0$ ,

in contradiction with the minimality of  $v$ ]. Therefore we may (and shall) assume that  $H(u_1) \ni u_1$ . We shall also assume

$$a < u_1 < c, fa$$

(and  $f^2u_1 \neq u_1$ , which will later be replaced by a stronger condition). There is  $u_2 \in [a, b]$  such that  $fu_2 = u_1$  and, since  $u_1 < fa$ , it follows that  $u_2$  is unique and satisfies  $c < u_2 < b$ . We have  $u_2 \in H(u_1)$  [because  $u_2 > c > u_1$  and  $fu_2 \in H(u_1)$ ] and if  $x \in H(u_1)$  then  $x \leq u_2$  [because  $x > u_2$  implies  $fx < u_1$ ]. Therefore,  $u_2$  is the maximum element of  $H(u_1)$ . Let

$$V_0 = \{x \in [a, b] : fx > u_2\}$$

then  $u_1 < V_0$  [because  $x \leq u_1$  implies  $fx \leq fu_1 \in H(u_1) \leq u_2$ ] and  $V_0 < u_2$  [because  $x \geq u_2$  implies  $fx \leq fu_2 = u_1 < u_2$ ]. Thus we may write  $V_0 = (v_1, v_2)$ , with  $u_1 < v_1 < c < v_2 < u_2$  [ $u_1 \neq v_1$  because  $f^2u_1 \neq u_1$ ]. We have  $v_1, v_2 \in H(u_1)$  [because  $v_1, v_2 > u_1$  and  $fv_1 = fv_2 = u_2 \in H(u_1)$ ].

Our assumptions ( $H(u_1) \ni u_1$ ,  $a < u_1 < c, fa$  and  $f^2u_1 \neq u_1$ ) and definitions give thus

$$H(u_1) \subset [u_1, v_1] \cup [v_2, u_2]$$

$$f[u_1, v_1] \subset [u_1, u_2] \quad , \quad f[v_2, u_2] = [u_1, u_2]$$

and

$$H(u_1) = \{x \in [u_1, u_2] : f^n x \notin V_0 \text{ for all } n \geq 0\} = fH(u_1)$$

Let us say that the open interval  $V_\alpha \subset [u_1, u_2]$  is of order  $n$  if  $f^n$  maps homeomorphically  $V_\alpha$  onto  $(v_1, v_2) = V_0$ . We have thus

$$H(u_1) = [u_1, u_2] \setminus \bigcup \text{ all } V_\alpha$$

By induction on  $n$  we shall see that

$$[u_1, u_2] \setminus \bigcup \text{ the } V_\alpha \text{ of order } \leq n$$

is composed of disjoint closed intervals  $J$ , such that  $f^n J \subset [u_1, v_1]$  or  $[v_2, u_2]$  when  $n > 0$ , and the endpoints of  $f^n J$  are  $u_1, u_2, v_1, v_2$  or an image of these points by  $f^k$  with  $k \leq n$ . Assume that the induction assumption holds for  $n$  (the case of  $n = 0$  is trivial) and let  $J$  be as indicated. Since  $f^n J \subset [u_1, v_1]$  or  $[v_2, u_2]$ ,  $f^{n+1}$  is monotone on  $J$ , and the endpoints of  $J$  are mapped by  $f^{n+1}$  outside of  $V_0$  [because  $u_1, u_2, v_1, v_2$  and their images by  $f^\ell$  are in  $H(u_1)$ , hence  $\notin (v_1, v_2)$ ]. The interval  $V_0$  is thus either inside of  $f^{n+1}J$  or disjoint from  $f^{n+1}J$ . Each  $V_\alpha$  of order  $n+1$  thus obtained is disjoint from other  $V_\alpha$  of order  $\leq n+1$ , and the closed intervals  $\tilde{J}$  in  $[u_1, u_2] \setminus \bigcup \text{ the } V_\alpha \text{ of order } \leq n+1$ , are such that the endpoints of  $f^{n+1}\tilde{J}$  are  $u_1, u_2, v_1, v_2$  or an image of these points by  $f^k$  with  $k \leq n+1$ , in agreement with our induction assumption.

We assume now that, for some  $N \geq 0$ , we have  $f^{N+1}u_1 = u_1$  (take  $N$  smallest with this property), and we assume also that  $(f^{N+1})'(u_1) > 0$ . [ $N = 0, 1$  cannot occur, in

particular  $f^2 u_1 \neq u_1$ . Thus  $N \geq 2$ , with  $f^N u_1 = u_2$ ,  $f^{N-1} u_1 \in \{v_1, v_2\}$ . Furthermore,  $(f^{N-1})'(u_1) < 0$  if  $f^{N-1} u_1 = v_1$ , and  $(f^{N-1})'(u_1) > 0$  if  $f^{N-1} u_1 = v_2$ , i.e.,  $f^{N-1}(u_1+) = v_1-$  or  $v_2+$ .

Using the above assumption we now show that none of the intervals  $J$  in

$$[u_1, u_2] \setminus \bigcup \text{ the } V_\alpha \text{ of order } \leq n$$

is reduced to a point. We proceed by induction on  $n$ , assuming that  $f^n J = [f^n x_1, f^n x_2]$ , where  $f^n x_1 < f^n x_2$  and  $f^n x_1$  is of the form  $v_2, u_1$  or  $f^\ell u_1$  with  $(f^\ell)'(u_1) > 0$  while  $f^n x_2$  is of the form  $v_1, u_2$  or  $f^\ell u_2$  with  $(f^\ell)'(u_2) > 0$ . Therefore the lower limit of  $f^{n+1} J$  is of the form  $f^m u_1$  with  $(f^m)'(u_1) > 0$  while the upper limit is of the form  $f^m u_2$  with  $(f^m)'(u_2) > 0$ . If

$$f^{n+1} J \supset (v_1, v_2)$$

so that a new  $V_\alpha$  of order  $n+1$  is created, the set  $f^{n+1} J \setminus (v_1, v_2)$  consists of two closed intervals, and one of them can be reduced to a point only if  $f^m u_1 = v_1$  with  $(f^m)'(u_1) > 0$  or if  $f^m u_2 = v_2$  with  $(f^m)'(u_2) > 0$ . So, either  $f^{m+2} u_1 = u_1$  with  $(f^{m+2})'(u_1) < 0$ , or  $f^{m+1} u_2 = u_2$  with  $(f^{m+1})'(u_2) < 0$  hence  $f^{m+1} u_1 = u_1$  with  $(f^{m+1})'(u_1) < 0$ , in contradiction with our assumption that  $(f^{N+1})'(u_1) > 0$ .

### 3 Consequences.

(No isolated points)

$H(u_1)$  is obtained from  $[u_1, u_2]$  by taking away successively intervals  $V_\alpha$  of increasing order. A given  $x \in H(u_1)$  will, at each step, belong to some small closed interval  $J$ , and the endpoints of  $J$  will not be removed in later steps, so that  $x$  cannot be an isolated point:  $H(u_1)$  has no isolated points.

(Markov property)

Our assumption  $f^{N+1} u_1 = u_1$  implies that, for  $n = 1, \dots, N-1$ , the point  $f^n u_1$  is one of the endpoints of an interval  $V_\alpha$  of order  $N-1-n$ , which we call  $V_{N-1-n}$ . These open intervals  $V_k$  are disjoint, and their complement in  $[u_1, u_2]$  consists of  $N$  intervals  $U_1, \dots, U_N$ . Each  $U_i$  is closed, nonempty, and not reduced to a point. Furthermore, each  $U_i$  (for  $i = 1, \dots, N$ ) is mapped by  $f$  homeomorphically to a union of intervals  $U_j$  and  $V_k$ : this is what we call *Markov property*.

We impose now the following condition:

### 4 Hyperbolicity.

There are constants  $A > 0, \alpha \in (0, 1)$  such that if  $x, fx, \dots, f^{n-1}x \in [u_1, v_1] \cup [v_2, u_2]$ , then

$$\left| \frac{d}{dx} f^n x \right|^{-1} < A \alpha^n$$

We label the intervals  $U_1, \dots, U_N$  from left to right, so that  $u_1$  is the lower endpoint of  $U_1$ , and  $u_2$  the upper endpoint of  $U_N$ . Define also an oriented graph with vertices  $U_j$

and edges  $U_j \rightarrow U_k$  when  $fU_j \supset U_k$ . Write  $U_{j_0} \xRightarrow{\ell} U_{j_\ell}$  if  $U_{j_0} \rightarrow U_{j_1} \rightarrow \dots \rightarrow U_{j_\ell}$ , and  $U_j \xRightarrow{\ell} U_k$  if  $U_j \xRightarrow{\ell} U_k$  for some  $\ell > 0$ .

**5 Lemma** (mixing).

- (a) For each  $U_j$  there is  $r \geq 0$  such that  $U_j \xRightarrow{r+3} U_1$ .
- (b) If there is  $s > 0$  such that  $U_1 \xRightarrow{s} U_1$  and  $U_1 \xRightarrow{s} U_N$ , then  $U_1 \xRightarrow{s} U_k$  for  $k = 1, \dots, N$ .
- (c) If there is  $s > 0$  such that  $U_j \xRightarrow{s} U_k$  for all  $U_j, U_k \in \{U_j : U_1 \xRightarrow{s} U_j \xRightarrow{s} U_1\}$ , then  $U_j \xRightarrow{s} U_k$  for all  $U_j, U_k \in \{U_1, \dots, U_N\}$ , and we say that  $H(u_1)$  is mixing.
- (d) In particular if  $N + 1$  is a prime, then  $H(u_1)$  is mixing.
- (e) Let  $u_1 < \tilde{u}_1 < c$ ,  $f a$ , and suppose that  $f^{\tilde{N}+1} \tilde{u}_1 = \tilde{u}_1$ ,  $(f^{\tilde{N}+1})'(u_1) > 0$ . Then if  $H(u_1)$  is mixing, so is  $H(\tilde{u}_1)$ .

(a) The interval  $U_j$  is contained in either  $[u_1, v_1]$  or  $[v_2, u_2]$ . Let the same hold for the successive images up to  $f^r U_j$ , but  $f^{r+1} U_j \ni c$  [hyperbolicity and the fact that  $U_j$  is not reduced to a point imply that  $r$  is finite]. Then  $U_j \xRightarrow{r+1} U_k$  with  $U_k \ni v_1$  or  $v_2$ , hence  $U_k \xRightarrow{2} U_1$  and  $U_j \xRightarrow{r+3} U_1$ .

(b) The  $U_j$  such that  $U_1 \xRightarrow{s} U_j$  form a set of consecutive intervals and, since this set contains  $U_1$  and  $U_N$  by assumption, it contains all  $U_j$  for  $j = 1, \dots, N$ .

(c) By assumption,  $U_1 \xRightarrow{s} U_1$  and  $U_1 \xRightarrow{s} U_N$ , so that  $U_1 \xRightarrow{s} U_k$  for  $k = 1, \dots, N$  by (b). Therefore,  $\{U_j : U_1 \xRightarrow{s} U_j \xRightarrow{s} U_1\} = \{U_1, \dots, U_N\}$  by (a), and thus  $U_j \xRightarrow{s} U_k$  for all  $U_j, U_k \in \{U_1, \dots, U_N\}$ .

(d) The *transitive* set  $\{U_j : U_1 \xRightarrow{s} U_j \xRightarrow{s} U_1\}$  decomposes into  $n$  disjoint subsets  $S_0, \dots, S_{n-1}$  such that  $S_0 \xRightarrow{1} S_1 \xRightarrow{1} \dots \xRightarrow{1} S_{n-1} \xRightarrow{1} S_0$  and there is  $s > 0$  such that  $U_j \xRightarrow{sn} U_k$  for all  $U_j, U_k \in S_m$ , where  $m = 0, \dots, n-1$ . We may suppose that  $U_1 \in S_0$ , and therefore if  $U_{(k)}$  denotes the interval containing  $f^k u_1$  we have  $U_{(k)} \in S_{(k)}$  where  $(k) = k \pmod n$ . Therefore  $N + 1$  is a multiple of  $n$ , where  $n \leq N < N + 1$ . In particular, if  $N + 1$  is prime, then  $n = 1$ , and  $U_j \xRightarrow{s} U_k$  for all  $U_j, U_k \in \{U_j : U_1 \xRightarrow{s} U_j \xRightarrow{s} U_1\}$ , so that (c) can be applied.

(e) Since  $H(\tilde{u}_1)$  is a compact subset of  $H(u_1)$ , without isolated points, the fact that  $H(u_1)$  is mixing implies that  $H(\tilde{u}_1)$  is mixing.  $\square$

## 6 Horseshoes.

Note that we have

$$H(u_1) = \{x \in [u_1, u_2] : f^n x \notin V_0 \text{ for all } n \geq 0\} = \cap_{n \geq 0} f^{-n}([u_1, u_2] \setminus V_0)$$

The sets  $U_i \cap H(u_1)$  form a *Markov partition* of  $H(u_1)$ , i.e.,  $f(U_i \cap H(u_1))$  is a finite union of sets  $U_j \cap H(u_1)$ .

A set  $H = H(u_1)$  as constructed in Section 2, with the hyperbolicity and mixing conditions will be called a *horseshoe*. A horseshoe is thus a mixing hyperbolic set with a Markov partition.

Remember that the open interval  $V_\alpha \subset [u_1, u_2]$  is of order  $n$  if  $f^n$  maps  $V_\alpha$  homeomorphically onto  $V_0 = (v_1, v_2)$ , and let  $|V_\alpha|$  be the length of  $V_\alpha$ . Hyperbolicity has the following consequence.

**7 Lemma** (a consequence of hyperbolicity).

There are constants  $B > 0$ ,  $\beta \in (0, 1)$  such that

$$\sum_{\alpha: \text{order } V_\alpha = n} |V_\alpha| \leq B\beta^n$$

It suffices to prove that

$$\text{Lebesgue meas. } ([u_1, u_2] \setminus \cup \text{ the } V_\alpha \text{ of order } \leq n) \leq G\beta^n$$

[incidentally, this shows that  $H(u_1)$  has Lebesgue measure 0].

Let  $J$  denote one of the closed intervals in

$$[u_1, u_2] \setminus \cup \text{ the } V_\alpha \text{ of order } \leq n$$

and suppose that  $J$  is one of the two intervals adjacent to a given  $V_\alpha$  of order  $n$ . There is  $n' > n$  such that  $J$  contains no interval  $V$  of order  $< n'$ , but  $J \supset V_{\alpha'}$  of order  $n'$ . We write  $J = J_{nn'}(V_\alpha, V_{\alpha'})$  and note that  $J$  is entirely determined by  $V_\alpha$  and  $V_{\alpha'}$  (of orders  $n, n'$  respectively). The intervals in

$$[u_1, u_2] \setminus \cup \text{ the } V_\alpha \text{ of order } \leq n$$

are all the  $J_{n_1 n_2}$  with  $n_1 \leq n$  and  $n_2 > n$ . There is a graph  $\Gamma$  with vertices  $V_\alpha$  and oriented edges  $J_{nn'}(V_\alpha, V_{\alpha'})$  such that for each  $V_\alpha$  of order  $n$  two edges  $J_{nn_1}(V_\alpha, V_{\alpha_1})$  come out of  $V_\alpha$  and, if  $n > 0$ , one edge  $J_{n_0 n}(V_{\alpha_0}, V_\alpha)$  goes in. The graph  $\Gamma$  is a tree, rooted at  $V_0$ .

We want to show that

$$\sum_{n_1 \leq n, n_2 > n} \sum_{\alpha_1 \alpha_2} |J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})| \leq G\beta^n$$

In order to do this we shall introduce intervals  $\tilde{J}_{n_1 n_2}^n(V_{\alpha_1}, V_{\alpha_2}) \supset J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$  such that, for fixed  $n$ , the  $\tilde{J}_{n_1 n_2}^n(V_{\alpha_1}, V_{\alpha_2})$  are disjoint, and we shall find  $\theta \in (0, 1)$  and an integer  $N > 0$  such that

$$\sum_{n_1 \leq n, n_2 > n} |\tilde{J}_{n_1+2N, n_2+2N}^{n+2N}(V_{\alpha'_1}, V_{\alpha'_2})| \leq \theta \sum_{n_1 \leq n, n_2 > n} |\tilde{J}_{n_1 n_2}^n(V_{\alpha_1}, V_{\alpha_2})|$$

(where sums over  $\alpha'_1, \alpha'_2$  and  $\alpha_1, \alpha_2$  are implied). In fact, we shall prove that

$$\sum^* |\tilde{J}_{n'_1, n'_2}^{n+2N}(V_{\alpha'_1}, V_{\alpha'_2})| \leq \theta |\tilde{J}_{n_1 n_2}^n(V_{\alpha_1}, V_{\alpha_2})| \quad (*)$$

for fixed  $\tilde{J}_{n_1 n_2}^n(V_{\alpha_1}, V_{\alpha_2})$  such that  $n_1 \leq n, n_2 > n$ , where the sum  $\sum^*$  extends over all  $\tilde{J}_{n'_1, n'_2}^{n+2N}(V_{\alpha'_1}, V_{\alpha'_2})$  such that  $J_{n'_1, n'_2}^n(V_{\alpha'_1}, V_{\alpha'_2})$  is above  $J_{n_1 n_2}^n(V_{\alpha_1}, V_{\alpha_2})$  in the tree  $\Gamma$ , and that  $n'_1 \leq n + 2N, n'_2 > n + 2N$ . [This means that  $\sum^*$  extends over  $\tilde{J}^{n+2N}$  corresponding to the closed intervals  $J^*$  of

$$[u_1, u_2] \setminus \cup \text{the } V_{\alpha'} \text{ of order } \leq n + 2N$$

such that  $J^* \subset J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$ .

Note that  $J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2}) \supset V_{\alpha_2}$  and that for some constant  $K_1$  independent of  $n_1, n_2$  we may write  $|J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})| \leq K_1 |V_{\alpha_2}|$  [otherwise  $J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$  would contain a  $V_{\alpha}$  of order  $< n_2$ ]. We can also compare  $|V_{\alpha_1}|$  and  $|V_{\alpha_2}|$  because  $f^{n_1} V_{\alpha_1} = f^{n_2} V_{\alpha_2} = V_0$ : using hyperbolicity and the smoothness of  $f$  we find a constant  $K_2$  such that  $|V_{\alpha_2}| \leq K_2 \alpha^{n_2 - n_1} |V_{\alpha_1}|$ . Thus

$$|J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})| \leq K_1 K_2 \alpha^{n_2 - n_1} |V_{\alpha_1}| \leq \alpha^{n_2 - n_1 - N} \frac{1}{3} |V_{\alpha_1}|$$

for suitable  $N$ . We also assume that  $2\alpha^N < 1$ .

If  $n_2 - n_1 < 2N$  we define  $\tilde{J}_{n_1 n_2}^n(V_{\alpha_1}, V_{\alpha_2}) = J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$ . If  $n_2 - n_1 \geq 2N$  we define  $\tilde{J}_{n_1 n_2}^n(V_{\alpha_1}, V_{\alpha_2})$  as the union of  $J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$  and an adjacent subinterval  $\tilde{V} \subset V_{\alpha_1}$  such that  $|\tilde{V}| = \alpha^{\frac{1}{2}(n - n_1)} \frac{1}{3} |V_{\alpha_1}|$  and therefore (since  $n < n_2$ )

$$|\tilde{V}| > \alpha^{\frac{1}{2}(n_2 - n_1)} \frac{1}{3} |V_{\alpha_1}| > \alpha^{n_2 - n_1 - N} \frac{1}{3} |V_{\alpha_1}| \geq |J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})|$$

If  $n + 2N < n_2$ , there is only one term in the left-hand side of (\*), and this term is  $\tilde{J}_{n_1 n_2}^{n+2N}(V_{\alpha_1}, V_{\alpha_2})$ , so that

$$\begin{aligned} \left| \frac{\tilde{J}_{n_1 n_2}^{n+2N}(V_{\alpha_1}, V_{\alpha_2})}{\tilde{J}_{n_1 n_2}^n(V_{\alpha_1}, V_{\alpha_2})} \right| &\leq \frac{\alpha^{\frac{1}{2}(n - n_1 + 2N)} \frac{1}{3} |V_{\alpha_1}| + \alpha^{n_2 - n_1 - N} \frac{1}{3} |V_{\alpha_1}|}{\alpha^{\frac{1}{2}(n - n_1)} \frac{1}{3} |V_{\alpha_1}|} \\ &= \alpha^N + \alpha^{n_2 - \frac{1}{2}n_1 - \frac{1}{2}n - N} \leq \alpha^N + \alpha^{n_2 - n - N} \leq 2\alpha^N \end{aligned}$$

If  $n + 2N \geq n_2$  there are several terms in the left-hand side of (\*), obtained from the interval  $J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$  from which at least a subinterval of length  $\frac{1}{3} |V_{\alpha_2}|$  has been taken out. Therefore

$$\sum^* \leq |J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})| - \frac{1}{3} |V_{\alpha_2}|$$

and

$$\frac{\sum^*}{|\tilde{J}_{n_1 n_2}^n(V_{\alpha_1}, V_{\alpha_2})|} \leq 1 - \frac{\frac{1}{3} |V_{\alpha_2}|}{|J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})|} \leq 1 - \frac{\frac{1}{3} |V_{\alpha_2}|}{K_1 |V_{\alpha_2}|} \leq 1 - \frac{1}{3K_1}$$



We have thus proved (\*) with  $\theta = \max(2\alpha^N, 1 - 1/3K_1)$ , and the lemma follows, with  $\beta^N = \theta$ .  $\square$

**8 Remark** (the set  $\tilde{H}$ ).

Starting from the horseshoe  $H = H(u_1)$  we can, by increasing  $u_1$  to  $\tilde{u}_1$  such that  $\tilde{u}_1 < c, fa$ , obtain a set  $\tilde{H} = H(\tilde{u}_1) \subset H$  such that  $\tilde{u}_1 \in \tilde{H}$  and the distance of  $\tilde{H}$  to  $\{u_1, u_2, v_1, v_2\}$  is  $\geq \epsilon > 0$ . [In fact, using our hyperbolicity assumption we can arrange that there is  $\tilde{N}$  such that  $f^{\tilde{N}+1}\tilde{u}_1 = \tilde{u}_1$ ,  $(f^{\tilde{N}+1})'(\tilde{u}_1) > 0$ . In that case  $\tilde{H}$  is mixing (Lemma 5(e)) and therefore again a horseshoe].

**9 Theorem.**

Let  $H = H(u_1)$  be a horseshoe, suppose that  $fa = f^2b \in H$ , and that  $\{f^n b : n \geq 0\}$  has a distance  $\geq \epsilon > 0$  from  $\{u_1, u_2, v_1, v_2\}$ . Then  $f$  has a unique a.c.i.m.  $\rho(x) dx$ . Furthermore

$$\rho(x) = \phi(x) + \sum_{n=0}^{\infty} C_n \psi_n(x)$$

The function  $\phi$  is continuous on  $[a, b]$ , with  $\phi(a) = \phi(b) = 0$ . For  $n \geq 0$  we shall choose  $w_n \in \{u_1, u_2, v_1, v_2\}$  with  $(w_n - c)(c - f^n b) < 0$  and let  $\theta_n$  be the characteristic function of  $\{x : (w_n - x)(x - f^n b) > 0\}$ . Then, the above constants  $C_n$  and spikes  $\psi_n$  are defined by

$$C_n = \phi(c) \left| \frac{1}{2} f''(c) \prod_{k=0}^{n-1} f'(f^k b) \right|^{-1/2}$$

$$\psi_n(x) = \frac{w_n - x}{w_n - f^n b} \cdot |x - f^n b|^{-1/2} \theta_n(x)$$

[The condition that  $\{f^n b : n \geq 0\}$  has distance  $\geq \epsilon$  from  $\{u_1, u_2, v_1, v_2\}$  is achieved, according to Remark 8, by taking  $\epsilon \leq |u_1 - a|, |u_2 - b|$ , and  $f^2 b \in \tilde{H}$ . Note also that  $\psi_n(c) = 0$ , so that  $\phi(c) = \rho(c)$ . Other choices of  $\psi_n$  can be useful, with the same singularity at  $f^n b$ , but greater smoothness at  $w_n$  and/or satisfying  $\int dx \psi_n(x) = 0$ ].

**10 Analysis.**

We analyze the problem before starting the proof. Near  $c$  we have

$$y = fx = b - A(x - c)^2 + \text{h.o.}$$

with  $A = -f''(c)/2 > 0$ , hence  $x - c = \pm((b - y)/A)^{1/2} + O(b - y)$ . Therefore, writing  $U = \rho(c)/\sqrt{A}$ , the density of the image  $f(\rho(x)dx)$  by  $f$  of  $\rho(x)dx$  has, near  $b$ , a singularity

$$\frac{U}{\sqrt{(b - x)}} + O(\sqrt{b - x})$$

and, near  $a$ , a singularity

$$\frac{U}{\sqrt{-f'(b)(x-a)}} + O(\sqrt{x-a})$$

To deal with the general case of the singularity at  $f^n b$ , define  $s_n = -\text{sgn} \prod_{k=0}^{n-1} f'(f^k b)$ , so that

$$\prod_{k=0}^{n-1} f'(f^k b) = -s_n U^2 C_n^{-2}$$

The density of  $f(\rho(x)dx)$  has then, near  $f^n b$ , a singularity given when  $s_n(x - f^n b) > 0$  by

$$\begin{aligned} & \frac{U}{\sqrt{(\prod_{k=0}^{n-1} |f'(f^k b)|)|x - f^n b|}} + O(\sqrt{|x - f^n b|}) \\ &= \frac{U}{\sqrt{-(x - f^n b) \prod_{k=0}^{n-1} f'(f^k b)}} + O(\sqrt{|x - f^n b|}) \\ &= \frac{C_n}{\sqrt{s_n(x - f^n b)}} + O(\sqrt{s_n(x - f^n b)}) \end{aligned}$$

and by 0 when  $s_n(x - f^n b) < 0$ .

We let now  $w_0 = u_2$  and, for  $n \geq 0$ , define  $w_{n+1} \in \{u_1, u_2, v_1, v_2\}$  inductively by:

$$(w_{n+1} - c)(f^{n+1}b - c) > 0 \quad , \quad (w_{n+1} - f^{n+1}b)(fw_n - f^{n+1}b) > 0$$

We have thus  $w_0 = u_2, w_1 = u_1$ , and in general

$$w_n \in \{u_1, u_2, v_1, v_2\} \quad , \quad (w_n - c)(f^n b - c) > 0 \quad , \quad s_n(w_n - f^n b) > 0 \quad , \quad |w_n - f^n b| \geq \epsilon$$

The above considerations show that the singularity expected near  $f^n b$  for the density of  $f(\rho(x)dx)$  is also represented by

$$\begin{aligned} & \left(1 - \frac{x - f^n b}{w_n - f^n b}\right) \cdot \frac{C_n}{\sqrt{s_n(x - f^n b)}} \theta_n(x) \\ &= C_n \frac{w_n - x}{w_n - f^n b} |x - f^n b|^{-1/2} \theta_n(x) = C_n \psi_n(x) \end{aligned}$$

in agreement with the claim of the theorem.

## 11 Lemma.

Write

$$f(\psi_n(x)dx) = \tilde{\psi}_{n+1}(x)dx \quad , \quad \tilde{\psi}_{n+1} = |f'(f^n b)|^{-1/2} \psi_{n+1} + \chi_n$$

Then, for  $n \geq 0$ , the  $\chi_n$  are continuous of bounded variation on  $[a, b]$ , with  $\chi_n(a) = \chi_n(b) = 0$ , and the  $\text{Var } \chi_n = \int_a^b |d\chi_n/dx| dx$  are bounded uniformly with respect to  $n$ . Furthermore, if  $n \geq 1$  and  $V_\alpha \subset \text{supp } \chi_n$ , then  $\chi_n|_{V_\alpha}$  extends to a holomorphic function  $\chi_{n\alpha}$  in a complex neighborhood  $D_\alpha$  of the closure of  $V_\alpha$  in  $\mathbf{R}$  (further specified in Section 12), with the  $|\chi_{n\alpha}|$  uniformly bounded.

The case  $n = 0$  can be handled by inspection, and we shall assume  $n \geq 1$ . We let

$$I_n = \begin{cases} (fa, b) & \text{if } f^n b \in [a, c) \\ (a, b) & \text{if } f^n b \in (c, b) \end{cases}$$

And define  $f_n^{-1} : I_n \mapsto (a, b)$  to be the inverse of  $f$  restricted respectively to  $(a, c)$  or  $(c, b)$  in the two cases above. We have then

$$\tilde{\psi}_{n+1}(x) = \frac{\psi_n(f_n^{-1}x)}{|f'(f_n^{-1}x)|}$$

Since  $n \geq 1$ , the region of interest  $f \text{supp } \psi_n \cup \text{supp } \psi_{n+1}$  is  $\subset [u_1, u_2] \subset (a, b)$ , and we have

$$f_n^{-1}x - f^n b = (x - f^{n+1}b)A_n(x)$$

where  $A_n$  is real analytic and  $A_n(f^{n+1}b) = (f'(f^n b))^{-1}$ . Therefore we may write

$$\frac{1}{f_n^{-1}x - f^n b} = \frac{f'(f^n b)}{x - f^{n+1}b} (1 + (x - f^{n+1}b)\tilde{A}_n(x))$$

$$\frac{1}{f'(f_n^{-1}x)} = \frac{1}{f'(f^n b)} (1 + (x - f^{n+1}b)\tilde{B}_n(x))$$

$$\frac{w_n - f_n^{-1}x}{w_n - f^n b} = 1 + (x - f^{n+1}b)\tilde{C}_n(x)$$

and since

$$\psi_n(f_n^{-1}x) = \theta_n(f_n^{-1}x) \left| \frac{w_n - f_n^{-1}x}{w_n - f^n b} \right| \cdot |f_n^{-1}x - f^n b|^{-1/2}$$

we find

$$\tilde{\psi}_{n+1}(x) = \frac{\theta_n(f_n^{-1}x) |f'(f^n b)|^{-1/2}}{\sqrt{|x - f^{n+1}b|}} (1 + (x - f^{n+1}b)\tilde{D}_n(x))$$

with  $\tilde{D}_n$  real analytic. Note that  $\tilde{\psi}_{n+1}$  and  $|f'(f^n b)|^{-1/2}\psi_{n+1}$  have the same singularity at  $f^{n+1}b$ . It follows readily that  $\psi_{n+1} - |f'(f^n b)|^{-1/2}\psi_{n+1}$  is a continuous function  $\chi_n$  vanishing at the endpoints of its support, and bounded uniformly with respect to  $n$ . It is easy to see that  $\text{Var } \chi_n$  is bounded uniformly in  $n$ . The extension of  $\chi_n|_{V_\alpha}$  to holomorphic  $\chi_{n\alpha}$  in  $D_\alpha$  is also handled readily (see Section 12 for the description of the  $D_\alpha$ ).  $\square$

## 12 The operator $\mathcal{L}$ and the space $\mathcal{A}$ .

We have  $f(\rho(x) dx) = (\mathcal{L}_{(1)}\rho)(x) dx$ , where the transfer operator  $\mathcal{L}_{(1)}$  on  $L^1(a, b)$  is defined by

$$\mathcal{L}_{(1)}\rho = \sum_{\pm} \frac{\rho \circ f_{\pm}^{-1}}{|f' \circ f_{\pm}^{-1}|}$$

and we have denoted by

$$f_{-}^{-1} : [fa, b] \mapsto [a, c] \quad \text{and} \quad f_{+}^{-1} [a, b] \mapsto [c, b]$$

the branches of the inverse of  $f$ . The invariance of  $\rho(x) dx$  under  $f$  is thus expressed by

$$\rho = \mathcal{L}_{(1)}\rho$$

We shall look for a solution of this equation in a Banach space  $\mathcal{A}$  defined below. Roughly speaking,  $\mathcal{A}$  consists of functions

$$\phi + \sum_{n=0}^{\infty} c_n \psi_n$$

where the  $\psi_n$  are defined in the statement of Theorem 9, and  $\phi : [a, b] \rightarrow \mathbf{C}$  is a less singular rest with certain analyticity properties.

Remember that we may write

$$[a, b] = H \cup [a, u_1) \cup (u_2, b] \cup \text{the } V_{\alpha} \text{ of all orders } \geq 0$$

We have (see Remark 8)

$$\text{clos } [a, u_1) \subset [a, \tilde{u}_1) \quad , \quad \text{clos } (u_2, b] \subset (\tilde{u}_2, b] \quad , \quad \text{clos } V_0 \subset \tilde{V}_0$$

where  $\tilde{u}_2$  and  $\tilde{V}_0 = (\tilde{v}_1, \tilde{v}_2)$ , are defined for  $\tilde{H}$  as  $u_2$  and  $V_0$  were defined for  $H$ . It is convenient to define  $V_{-1} = (u_2, b]$  and  $V_{-2} = [a, u_1)$  (of order  $-1$  and  $-2$  respectively) so that

$$[a, b] = H \cup \text{the } V_{\alpha} \text{ of all orders } \geq -2$$

We also define  $\tilde{V}_{-1} = (\tilde{u}_2, b]$ ,  $\tilde{V}_{-2} = [a, \tilde{u}_1)$ . We let now  $\tilde{V}_{\alpha}$  denote the unique interval in  $[a, b] \setminus \tilde{H}$  such that  $V_{\alpha} \subset \tilde{V}_{\alpha}$ . Note that the map  $V_{\alpha} \mapsto \tilde{V}_{\alpha}$  is not injective!

For each  $V_{\alpha}$  of order  $\geq 0$  we may choose an open set  $D_{\alpha} \subset \mathbf{C}$  such that

$$\tilde{V}_{\alpha} \supset D_{\alpha} \cap \mathbf{R} \supset \text{clos } V_{\alpha}$$

and, if  $fV_{\beta} = V_{\alpha}$  of order  $\geq 0$ ,  $fD_{\beta} \supset \text{clos } D_{\alpha}$  [we have here denoted by  $\text{clos } V_{\alpha}$  the closure of  $V_{\alpha}$  in  $\mathbf{R}$ , and by  $\text{clos } D_{\alpha}$  the closure of  $D_{\alpha}$  in  $\mathbf{C}$ ]. Let also  $R_a, R_b$  be two-sheeted Riemann surfaces, branched respectively at  $a, b$ , with natural projections  $\pi_a, \pi_b : R_a, R_b \rightarrow \mathbf{C}$ . We may choose open sets  $D_{-1}, D_{-2} \subset \mathbf{C}$  such that, for  $\alpha = -1, -2$ ,

$$\tilde{V}_{\alpha} \supset D_{\alpha} \cap \{x \in \mathbf{R} : a \leq x \leq b\} \supset \text{clos } V_{\alpha}$$

and  $f$  extends to holomorphic maps  $\tilde{f}_{-1} : D_0 \rightarrow R_b, \tilde{f}_{-2} : (\tilde{f}_{-1}D_0) \rightarrow R_a$  such that  $\tilde{f}_{-1}D_0 \supset \pi_b^{-1}\text{clos } D_{-1}, \tilde{f}_{-2}\pi_b^{-1}D_{-1} \supset \pi_a^{-1}\text{clos } D_{-2}$ . [We shall say that  $\tilde{f}_{-1}$  sends  $(v_1, c)$  to the *upper* sheet of  $R_b$  and  $(c, v_2)$  to the *lower* sheet of  $R_b$ ;  $\tilde{f}_{-2}$  sends the upper (lower) sheet of  $R_b$  to the upper (lower) sheet of  $R_a$ ].

We come now to a precise definition of the complex Banach space  $\mathcal{A}$ . We write  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$  where the elements of  $\mathcal{A}_1$  are of the form  $(\phi_\alpha)$  and the elements of  $\mathcal{A}_2$  of the form  $(c_n)$ . Here the index set of the  $\phi_\alpha$  is the same as the index set of the intervals  $V_\alpha$  (of order  $\geq -2$ ); the index  $n$  of the  $c_n \in \mathbf{C}$  takes the values  $0, 1, \dots$  [the  $c_n$  should not be confused with the critical point  $c$ ]. We assume that  $\phi_\alpha$  is a holomorphic function in  $D_\alpha$  when  $V_\alpha$  is of order  $\geq 0$ , while  $\phi_{-1}, \phi_{-2}$  are holomorphic on  $\pi_b^{-1}D_{-1}, \pi_a^{-1}D_{-2}$  and, for all  $\alpha$ ,  $\|\phi_\alpha\| = \sup_{z \in D_\alpha} |\phi_\alpha(z)| < \infty$ .

[We shall later consider a function  $\phi : [a, b] \rightarrow \mathbf{C}$  such that  $\phi|V_\alpha = \phi_\alpha|V_\alpha$  when  $V_\alpha$  is of order  $\geq 0$ . For  $x \in V_{-1}$  we shall require  $\phi(x) = \Delta\phi(x) = \phi_{-1}(x^+) - \phi_{-1}(x^-)$  where  $x^+(x^-)$  is the preimage of  $x$  by  $\pi_b$  on the upper (lower) sheet of  $\pi_b^{-1}D_{-1}$ ; for  $x \in V_{-2}$  we shall require  $\phi(x) = \Delta\phi_{-2}(x) = \phi_{-2}(x^+) - \phi_{-2}(x^-)$  where  $x^+(x^-)$  is the preimage of  $x$  by  $\pi_a$  on the upper (lower) sheet of  $\pi_a^{-1}D_{-2}$ . But at this point we discuss an operator  $\mathcal{L}$  on  $\mathcal{A}$  instead of the transfer operator  $\mathcal{L}_{(1)}$  acting on functions  $\phi + \sum_n c_n \psi_n$ ].

Let  $\gamma, \delta$  be such that  $1 < \gamma < \beta^{-1}, 1 < \delta < \alpha^{-1/2}$  with  $\beta$  as in Lemma 7 and  $\alpha$  as in the definition of hyperbolicity (Section 4). We write

$$\|(\phi_\alpha)\|_1 = \sup_{n \geq -2} \gamma^n \sum_{\alpha: \text{order } V_\alpha = n} |V_\alpha| \|\phi_\alpha\| \quad , \quad \|(c_n)\|_2 = \sup_{n \geq 0} \delta^n |c_n|$$

and, for  $\Phi = ((\phi_\alpha), (c_n))$ , we let  $\|\Phi\| = \|(\phi_\alpha)\|_1 + \|(c_n)\|_2$ . We let then  $\mathcal{A}_1, \mathcal{A}_2$  be the Banach spaces of sequences  $(\phi_\alpha), (c_n)$  as above, such that the norms  $\|(\phi_\alpha)\|_1, \|(c_n)\|_2$  are finite. We shall define  $\mathcal{L}$  on  $\mathcal{A}$  such that  $\mathcal{L}\Phi = \tilde{\Phi}$ . We first describe what contribution each  $\phi_\alpha$  or  $c_n$  gives to  $\tilde{\Phi}$  and then we shall check that this is a consistent description of an element  $\tilde{\Phi}$  of  $\mathcal{A}$ .

$$(i) \quad \phi_\beta \Rightarrow \hat{\phi}_{\beta\alpha} = \frac{\phi_\beta}{|f'|} \circ (f|D_\beta)^{-1} \quad \text{in } D_\alpha \text{ if order } \beta > 0 \text{ and } fV_\beta = V_\alpha$$

[we have here denoted by  $|f'|$  the holomorphic function  $\pm f'$  such that  $\pm f' > 0$  for real argument, we shall use the same notation in (ii)-(vi) below].

$$(ii) \quad \phi_0 \Rightarrow \left( \hat{c}_0 = C_0\phi_0(c), \hat{\phi}_{-1} = \pm \frac{\phi_0}{|f'|} \circ \tilde{f}_{-1}^{-1} - C_0\phi_0(c)(\pm \frac{1}{2}\psi_0 \circ \pi_b) \quad \text{in } \pi_b^{-1}D_{-1} \right)$$

where the signs  $\pm$  correspond to the upper/lower sheet of  $\pi_b^{-1}D_{-1}$ . We claim that  $\hat{\phi}_{-1}$  is holomorphic in  $\pi_b^{-1}D_{-1}$  as the difference of two meromorphic functions with a simple pole at the branch point  $b$ , with the same residue. To see this we uniformize  $\pi_b^{-1}D_{-1}$  by the

map  $u \mapsto b - u^2$ . We have thus to express  $\pm \frac{\phi_0}{|f'|}(c+x) = \frac{\phi_0}{f'}(c+x)$  in terms of  $u$  where

$c+x = \tilde{f}_{-1}^{-1}(b-u^2)$  or  $u = \sqrt{b - \tilde{f}_{-1}(c+x)}$  which gives a meromorphic function with a simple pole  $1/2\sqrt{A}u$ . Since  $\pm C_0\phi_0(c)\psi_0(b-u^2)$  is meromorphic with the same simple pole,  $\hat{\phi}_{-1}$  is holomorphic in  $\pi_b^{-1}D_{-1}$ .

$$(iii) \phi_{-1} \Rightarrow \hat{\phi}_{-2} = \frac{\phi_{-1}}{|f'|} \circ \tilde{f}_{-2}^{-1} \quad \text{in } \pi_a^{-1}D_{-2}.$$

$$(iv) \phi_{-2} \Rightarrow \hat{\phi}_\alpha = \frac{\Delta\phi_{-2}}{f'} \circ f^{-1} \quad \text{in } D_\alpha \text{ if } f(a, u_1) \supset V_\alpha, 0 \text{ otherwise}$$

[we have written  $\Delta\phi_{-2}(x) = \phi_{-2}(x^+) - \phi_{-2}(x^-)$  where  $x^+(x^-)$  is the preimage of  $x$  by  $\pi_a$  on the upper (lower) sheet of  $\pi_a^{-1}D_{-2}$ ].

$$(v) c_0 \Rightarrow \left( \hat{c}_1 = |f'(b)|^{-1/2}c_0, \chi_0 = \pm \frac{1}{2}c_0 \left( \frac{\psi_0}{|f'|} \circ \pi_b \circ \tilde{f}_{-2}^{-1} - |f'(b)|^{-1/2}\psi_1 \circ \pi_a \right) \right. \\ \left. \text{in } \pi_a^{-1}D_{-2} \right) \text{ where the sign } \pm \text{ corresponds to the upper/lower sheet of } \pi_a^{-1}D_{-2}.$$

$$(vi) c_n \Rightarrow \left( \hat{c}_{n+1} = |f'(f^n b)|^{-1/2}c_n, \chi_{n\alpha} = c_n \left[ \frac{\psi_n}{|f'|} \circ f_n^{-1} - |f'(f^n b)|^{-1/2}\psi_{n+1} \right] \right. \\ \left. \text{in } D_\alpha \text{ if } V_\alpha \subset \{x : \theta_n(f_n^{-1}x) > 0\}, 0 \text{ otherwise} \right)$$

if  $n \geq 1$ .

We may now write

$$\tilde{\Phi} = ((\tilde{\phi}_\alpha), (\tilde{c}_n))$$

where

$$\begin{aligned} \tilde{\phi}_{-2} &= \hat{\phi}_{-2} + \chi_0 && \text{(see (iii),(v))} \\ \tilde{\phi}_{-1} &= \hat{\phi}_{-1} && \text{(see(ii))} \\ \tilde{\phi}_\alpha &= \sum_{\beta: fV_\beta = V_\alpha} \hat{\phi}_{\beta\alpha} + \hat{\phi}_\alpha + \sum_{n \geq 1} \chi_{n\alpha} \text{ if order } \alpha \geq 0 && \text{(see (i),(iv),(vi))} \\ \tilde{c}_0 &= \hat{c}_0 && \text{(see (ii))} \\ \tilde{c}_1 &= \hat{c}_1 && \text{(see (v))} \\ \tilde{c}_n &= \hat{c}_n && \text{for } n > 1 \quad \text{(see (vi))} \end{aligned}$$

Note that, corresponding to the decomposition  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ , we have

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_0 + \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix}$$

where

$$\begin{aligned} \mathcal{L}_0(\phi_\alpha) &= (\sum_{\beta: fV_\beta = V_\alpha} \hat{\phi}_{\beta\alpha}) \\ \mathcal{L}_1(\phi_\alpha) &= (\hat{\phi}_\alpha) \\ \mathcal{L}_2(c_n) &= (\chi_0, (\sum_{n \geq 1} \chi_{n\alpha})_{\alpha > -1}) \\ \mathcal{L}_3(\phi_\alpha) &= (\hat{c}_0, (0)_{n > 0}) \\ \mathcal{L}_4(c_n) &= (0, (\hat{c}_n)_{n > 0}) \end{aligned}$$

Holomorphic functions in  $D_\alpha$  are defined by (i),(iv),(vi) when order  $\alpha \geq 0$ , and in  $\pi_b^{-1}D_{-1}$ ,  $\pi_a^{-1}D_{-2}$  by (ii),(iii),(v). Using Lemma 7, one sees that  $\mathcal{L}_0, \mathcal{L}_1$  are bounded  $\mathcal{A}_1 \rightarrow \mathcal{A}_1$ . Using Lemma 11, one sees that  $\mathcal{L}_3$  is bounded  $\mathcal{A}_2 \rightarrow \mathcal{A}_1$ . It is also readily seen that  $\mathcal{L}_2, \mathcal{L}_4$  are bounded, so that  $\mathcal{L} : \mathcal{A} \rightarrow \mathcal{A}$  is bounded.

**13 Theorem** (structure of  $\mathcal{L}$ ).

With our definitions and assumptions, the bounded operator  $\mathcal{L} : \mathcal{A} \rightarrow \mathcal{A}$  is a compact perturbation of  $\mathcal{L}_0 \oplus \mathcal{L}_4$ ; its essential spectral radius is  $\leq \max(\gamma^{-1}, \delta\alpha^{1/2})$ .

Since  $fa \in \tilde{H}$ , we may assume that  $f(a, u_1) \supset V_\alpha$  implies  $f(D_{-2} \setminus \text{negative reals}) \supset \text{clos } D_\alpha$ . Therefore,  $\phi_{-2} \mapsto \hat{\phi}_\alpha|_{D_\alpha}$  is compact. For  $N$  positive integer, define the operator  $\mathcal{L}_{N1}$  such that

$$\mathcal{L}_{N1}(\phi_\alpha) = \frac{\Delta\phi_{-2}}{f'} \circ f^{-1} \quad \text{in } D_\alpha \text{ if } f(a, u_1) \supset V_\alpha \text{ and order } \alpha > N, 0 \text{ otherwise}$$

Then  $\mathcal{L}_1$  is a perturbation of  $\mathcal{L}_{N1}$  by a compact operator and, using Lemma 7, we see that

$$\|\mathcal{L}_{N1}(\phi_\alpha)\|_1 \leq C \sup_{n > N} \gamma^n \beta^n \rightarrow 0 \quad \text{when } N \rightarrow \infty$$

We can write  $\mathcal{L}_2 = \mathcal{L}_{N2} + \text{finite range}$ , where

$$\mathcal{L}_{N2}(c_n) = (0, 0, (\sum_{n \geq N} \chi_{n\alpha})_{\alpha \geq 0})$$

Using Lemma 11 we find a bound  $\|\sum_{n \geq N} \chi_{n\alpha}\| \leq C'\delta^N$  and, using Lemma 7,

$$\|\mathcal{L}_{N2}\|_{\mathcal{A}_2 \rightarrow \mathcal{A}_1} \leq C''\delta^N \rightarrow 0 \quad \text{when } N \rightarrow \infty$$

The operator  $\mathcal{L}_3$  has one-dimensional range. Therefore  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$  are compact operators, and the essential spectral radius of  $\mathcal{L}$  is the max of the essential spectral radius of  $\mathcal{L}_0$  on  $\mathcal{A}_1$  and  $\mathcal{L}_4$  on  $\mathcal{A}_2$ .

The spectral radius of  $\mathcal{L}_4$  is

$$\leq \|\mathcal{L}_4^N\|^{1/N} \leq (\delta^N C''' \sup_{\ell \geq 0} \prod_{k=0}^{N-1} |f'(f^{k+\ell}b)|^{-1/2})^{1/N} \quad \text{with limit } < \delta\alpha^{1/2} \text{ when } N \rightarrow \infty$$

The essential spectral radius of  $\mathcal{L}_0$  is

$$\begin{aligned} &\leq \lim_{N \rightarrow \infty} \frac{\sup_{n \geq N} \gamma^n \sum_{\alpha: \text{order } V_\alpha = n} |V_\alpha| \cdot \|\sum_{\beta: fV_\beta = V_\alpha} \hat{\phi}_{\beta\alpha}\|}{\sup_{n \geq N} \gamma^{n+1} \sum_{\beta: \text{order } V_\beta = n+1} |V_\beta| \cdot \|\phi_\beta\|} \\ &\leq \gamma^{-1} \lim_{\text{order } V_\alpha \rightarrow \infty} \frac{|V_\alpha| \cdot \|\sum_{\beta: fV_\beta = V_\alpha} \hat{\phi}_{\beta\alpha}\|}{\sum_{\beta: fV_\beta = V_\alpha} |V_\beta| \cdot \|\phi_\beta\|} = \gamma^{-1} \end{aligned}$$

In fact, no eigenvalue of  $\mathcal{L}_0$  can be  $> \gamma^{-1}$ , so the spectral radius of  $\mathcal{L}_0$  acting on  $\mathcal{A}_1$  is  $\leq \gamma^{-1}$ . The essential spectral radius of  $\mathcal{L}$  is thus  $\leq \max(\gamma^{-1}, \delta\alpha^{1/2})$ .  $\square$

[Note also that when  $\gamma \rightarrow \beta^{-1}, \delta \rightarrow 1$ , we have  $\max(\gamma^{-1}, \delta\alpha^{1/2}) \rightarrow \max(\beta, \alpha^{1/2})$ ].

## 14 The eigenvalue 1 of $\mathcal{L}$ .

Let the map  $\Delta : \mathcal{A}_1 \rightarrow L^1(a, b)$  be such that  $\Delta(\phi_\alpha)|(a, u_1) = \Delta\phi_{-2}$ ,  $\Delta(\phi_\alpha)|(u_2, b) = \Delta\phi_{-1}$ , and  $\Delta(\phi_\alpha)|V_\beta = \phi_\beta$  if order  $\beta \geq 0$ . We also define  $w : \mathcal{A} \rightarrow L^1(a, b)$  by  $w((\phi_\alpha), (c_n)) = \Delta(\phi_\alpha) + \sum_{n=0}^{\infty} c_n \psi_n$  and check readily that

$$w\mathcal{L}\Phi = \mathcal{L}_{(1)}w\Phi$$

If  $\lambda^0 \neq 0$  is an eigenvalue of  $\mathcal{L}$ , and  $\Phi^0 = ((\phi_\alpha^0), (c_n^0))$  is an eigenvector to this eigenvalue, we have  $w\Phi^0 \neq 0$  [because  $w\Phi^0 = 0$  implies  $\phi_0^0 = 0$ , hence  $\phi_{-1}^0 = 0, \phi_{-2}^0 = 0$ , and  $(c_n^0) = 0$ ; then  $\Delta(\phi_\alpha^0) = 0$ , so  $\phi_\alpha^0 = 0$  when order  $\alpha \geq 0$ , *i.e.*,  $\Phi_0 = 0$ ]. Therefore

$$\lambda^0 w\Phi^0 = \mathcal{L}_{(1)}(w\Phi^0)$$

$$|\lambda^0| \int_a^b |w\Phi^0| = \int_a^b |\mathcal{L}_{(1)}(w\Phi^0)| \leq \int_a^b \mathcal{L}_{(1)}|w\Phi^0| = \int_a^b |w\Phi^0|$$

hence  $|\lambda^0| \leq 1$ .

If  $c_0^0 = 0$ , then  $(c_n^0) = 0$ , and  $\lambda^0$  is thus an eigenvalue of  $\mathcal{L}_0$  acting on  $\mathcal{A}_1$ , so that  $|\lambda^0| \leq \gamma^{-1}$  (see Section 13). Therefore  $|\lambda^0| > \gamma^{-1}$  implies  $c_0^0 \neq 0, c_1^0 \neq 0$ , hence  $\Delta\phi_{-1} + c_0\psi_0 \neq 0$ ,  $\Delta\phi_{-2} + c_1\psi_1 \neq 0$ . Note that, by analyticity,  $\Delta\phi_{-2} + c_1\psi_1$  is nonzero almost everywhere in  $(a, u_1)$ . The image  $f(a, u_1)$  contains some (small) interval  $U_{i_0} \cap f^{-1}(U_{i_1} \cap f^{-1}(U_{i_2} \dots))$  on which the image of  $\Delta\phi_{-2} + c_1\psi_1$  by  $\mathcal{L}_{(1)}$  does not vanish, and therefore (by mixing),

$$\int_a^b |\mathcal{L}_{(1)}w\Phi^0| < \int_a^b \mathcal{L}_{(1)}|w\Phi^0|$$

when  $w\Phi^0/|w\Phi^0|$  is not constant on  $(a, b)$ . Thus either (after multiplication of  $\Phi^0$  by a suitable constant  $\neq 0$ ),  $w\Phi^0 \geq 0$ , or

$$|\lambda^0| \int_a^b |w\Phi^0| < \int_a^b |w\Phi^0| \quad (*)$$

*i.e.*,  $|\lambda^0| < 1$ . Thus 1 is the only possible eigenvalue  $\lambda^0$  with  $|\lambda^0| = 1$ , but 1 is an eigenvalue, otherwise the spectral radius of  $\mathcal{L}$  would be  $< 1$  [contradicting the fact that  $\int_a^b w\mathcal{L}^n\Phi = \int_a^b w\Phi > 0$  when  $w\Phi > 0$ ]. (\*) also implies that if  $\mathcal{L}\Phi^1 = \Phi^1$ , then  $w\Phi^1$  is proportional to  $w\Phi^0$ , hence  $\phi_0^1$  is proportional to  $\phi_0^0$ , hence  $\Phi^1$  is proportional to  $\Phi^0$ . Furthermore, the generalized eigenspace to the eigenvalue 1 contains only the multiples of  $\Phi_0$  [otherwise there would exist  $\Phi^1$  such that  $\mathcal{L}^n\Phi^1 = \Phi^1 + n\Phi^0$ , contradicting  $\int_a^b w\mathcal{L}\Phi^1 = \int_a^b w\Phi^1$ ]. We have proved the first part of the following

## 15 Proposition.

(a) *Apart from the simple eigenvalue 1, the spectrum of  $\mathcal{L}$  has radius  $< 1$ . The eigenvector  $\Phi^0$  to the eigenvalue 1 (after multiplication by a suitable constant  $\neq 0$ ) satisfies  $w\Phi^0 \geq 0$ .*



(b) Write  $\Phi^0 = ((\phi_\alpha^0), (c_n^0))$  and  $\Delta(\phi_\alpha^0) = \phi^0$ , then  $\phi^0$  is continuous, of bounded variation, and  $\phi^0(a) = \phi^0(b) = 0$ .

The interval  $[u_1, u_2]$  is divided into  $N$  closed intervals  $W_1, \dots, W_N$  by the points  $f^n u_1$  for  $n = 1, \dots, N-1$ . The intervals  $W_1, \dots, W_N$  are ordered from left to right, by doubling the common endpoints we make the  $W_j$  disjoint. Define  $\gamma^0 = (\gamma_j^0)_{j=1}^N$  by  $\gamma_j^0 = \phi^0|_{W_j} \in L^1(W_j)$ . Then, the equation  $\Phi^0 = \mathcal{L}\Phi^0$  implies

$$\gamma^0 = \mathcal{L}_* \gamma^0 + \eta \quad (*)$$

or

$$\gamma_j^0 = \sum_k \mathcal{L}_{jk} \gamma_k^0 + \eta_j$$

where  $\mathcal{L} = (\mathcal{L}_{jk})$  is a transfer operator defined as follows. Letting  $(f^{-1})_{kj} : W_j \rightarrow W_k$  be such that  $f \circ (f^{-1})_{kj}$  is the identity on  $W_j$  we write

$$\mathcal{L}_{jk} \gamma_k = \begin{cases} \frac{\gamma_k \circ (f^{-1})_{kj}}{|f' \circ (f^{-1})_{kj}|} & \text{if } fW_k \supset W_j \\ 0 & \text{otherwise} \end{cases}$$

[the term  $\mathcal{L}_* \gamma^0$  in  $(*)$  comes from (i) in Section 12]. We let

$$\eta_j = \sum_{n=0}^{\infty} \eta_{jn}$$

Here

$$\eta_{j0}(x) = \frac{\Delta \phi_{-2}^0(y)}{f'(y)}$$

if  $f(a, u_1) \cap W_j$  contains more than one point, and  $y \in (a, u_1)$ ,  $fy = x \in W_j$ ; we let  $\eta_{j0}(x) = 0$  otherwise [this term comes from (iv) in Section 12]. For  $n \geq 1$ , we let  $\eta_{jn} = C_n \chi_n|_{W_j}$  where  $\chi_n = (\psi_n/|f'|) \circ f_n^{-1} - |f'(f^n b)|^{-1/2} \psi_{n+1}$  [this term comes from (vi) in Section 12].

Because  $f u_1$  is one of the division points between the intervals  $W_j$ , the function  $\eta_{j0}$  is continuous on  $W_j$ ; the  $\eta_{jn}$  for  $n \geq 1$  are also continuous. Furthermore,  $\eta_{j0}$  and the  $\eta_{jn}$  for  $n \geq 1$  are uniformly of bounded variation. If  $\mathcal{H}_j$  denotes the Banach space of continuous functions of bounded variation on  $W_j$  we have thus  $\eta_j \in \mathcal{H}_j$  for  $j = 1, \dots, N$ . We shall now obtain an upper bound on the essential spectral radius of  $\mathcal{L}_*$  acting on  $\mathcal{H} = \oplus_1^N \mathcal{H}_j$  by studying  $\|\mathcal{L}_*^n - F_n\|$ , where  $F_n$  has finite-dimensional range (we use here a simple case of an argument due to Baladi and Keller [4]). Define

$$W_{i_n \dots i_0} = \{x \in W_{i_n} : fx \in W_{i_{n-1}}, \dots, f^n x \in W_{i_0}\}$$

when  $fW_{i_k} \supset W_{i_{k-1}}$  for  $k = n, \dots, 1$ . For  $\eta = (\eta_j) \in \mathcal{H}$ , we let  $\pi_n \eta = (\pi_{jn} \eta_j)$  where  $\pi_{jn} \eta_j$  is a piecewise affine function on  $W_j$  such that  $(\pi_{jn} \eta_j)(x) = \eta_j(x)$  whenever  $x$  is an endpoint of  $W_j$  or of an interval  $W_{ji_{n-1} \dots i_0}$ , and is affine between all such endpoints. Then  $F_n = \mathcal{L}_*^n \pi_n$  has finite rank (i.e., finite-dimensional range), and  $\mathcal{L}_*^n - F_n = \mathcal{L}_*^n (1 - \pi_n)$  maps

$\mathcal{H}$  to  $\mathcal{H}$ . Let  $\text{Var } \gamma = \sum_1^N \text{Var}_j \gamma_j$  where  $\text{Var}_j$  is the total variation on  $W_j$ . Let also  $\|\cdot\|_0$  denote the sup-norm and  $\|\cdot\| = \max\{\text{Var } \cdot, \|\cdot\|_0\}$  be the bounded variation norm. We have

$$\text{Var}(\gamma - \pi_n \gamma) \leq 2\text{Var } \gamma$$

$$\sum_{i_0 \cdots i_n} \|(\gamma - \pi_n \gamma)|_{W_{i_n \cdots i_0}}\|_0 \leq \text{Var } \gamma$$

[the second inequality follows from the first because  $\gamma - \pi_n \gamma$  vanishes at the endpoints of  $W_{i_n \cdots i_0}$ ]. Since  $\mathcal{L}_*^n(1 - \pi_n)\gamma$  vanishes at the endpoints of the  $W_j$ , we have

$$\begin{aligned} \|(\mathcal{L}_*^n - F_n)\gamma\| &= \text{Var}((\mathcal{L}_*^n - F_n)\gamma) \\ &= \text{Var} \sum_{i_0 \cdots i_n} ((\gamma - \pi_n \gamma)_{i_n} \circ \tilde{f}_{i_n \cdots i_0})(\tilde{f}' \circ \tilde{f}_{i_n \cdots i_0}) \cdots (\tilde{f}' \circ \tilde{f}_{i_1 i_0}) \end{aligned}$$

where we have written

$$\tilde{f}_{i_\ell \cdots i_0} = (f^{-1})_{i_\ell i_{\ell-1}} \circ \cdots \circ (f^{-1})_{i_1 i_0}$$

and

$$\tilde{f}' = \frac{1}{|f'|}$$

hence

$$\begin{aligned} \|(\mathcal{L}_*^n - F_n)\gamma\| &\leq \sum_{i_0 \cdots i_n} \text{Var}[(\gamma - \pi_n \gamma)_{i_n} \circ \tilde{f}_{i_n \cdots i_0})(\tilde{f}' \circ \tilde{f}_{i_n \cdots i_0}) \cdots (\tilde{f}' \circ \tilde{f}_{i_1 i_0})] \\ &= \sum_{i_0 \cdots i_n} \text{Var}[(\gamma - \pi_n \gamma)|_{W_{i_n \cdots i_0}} \prod_{\ell=0}^{n-1} (\tilde{f}' \circ (f^\ell|_{W_{i_n \cdots i_0}}))] \end{aligned}$$

The right-hand side is bounded by a sum of  $n+1$  terms where  $\text{Var}$  is applied to  $(\gamma - \pi_n \gamma)|_{W_{i_n \cdots i_0}}$  or a factor  $\tilde{f}' \circ (f^\ell|_{W_{i_n \cdots i_0}})$ , and the other factors are bounded by their  $\|\cdot\|_0$ -norm. Thus, using the hyperbolicity condition of Section 4, we have

$$\begin{aligned} &\|(\mathcal{L}_*^n - F_n)\gamma\| \\ &\leq \text{Var}(\gamma - \pi_n \gamma) \cdot A\alpha^n + \sum_{\ell=0}^{n-1} \sum_{i_0 \cdots i_n} \|(\gamma - \pi_n \gamma)|_{W_{i_n \cdots i_0}}\|_0 \cdot A\alpha^\ell \cdot \text{Var}(\tilde{f}'|_{W_{i_n \cdots i_0}}) \cdot A\alpha^{n-\ell-1} \\ &\leq 2A\alpha^n \text{Var } \gamma + nA^2\alpha^{n-1} \text{Var } \tilde{f}' \sum_{i_0 \cdots i_n} \|(\gamma - \pi_n \gamma)|_{W_{i_n \cdots i_0}}\|_0 \\ &\leq (2A + nA^2\alpha^{-1} \text{Var } \tilde{f}')\alpha^n \text{Var } \gamma \leq (2A + nA^2\alpha^{-1} \text{Var } \tilde{f}')\alpha^n \|\gamma\| \end{aligned}$$

so that

$$\|\mathcal{L}_*^n - F_n\| \leq (2A + nA^2\alpha^{-1} \text{Var } \tilde{f}')\alpha^n$$

and therefore  $\mathcal{L}_*$  has essential spectral radius  $\leq \alpha < 1$  on  $\mathcal{H}$ . Suppose that there existed an eigenfunction  $\gamma \in \mathcal{H}$  to the eigenvalue 1 of  $\mathcal{L}_*$ ; the fact that  $\gamma$  is continuous and  $\neq 0$  on some  $W_j$  would imply

$$\int (\mathcal{L}_*^n |\gamma|)(x) dx < \int |\gamma|(x) dx$$

[because, for some  $n$ ,  $\mathcal{L}_*^n$  sends "mass" into  $V_0$ ]. But this is in contradiction with

$$\int |\gamma|(x) dx = \int |\mathcal{L}_*^n \gamma|(x) dx \leq \int (\mathcal{L}_*^n |\gamma|)(x) dx$$

Therefore, 1 cannot be an eigenvalue of  $\mathcal{L}_*$ , and there is  $\gamma = (1 - \mathcal{L}_*)^{-1} \eta \in H$  such that

$$\gamma = \mathcal{L}_* \gamma + \eta$$

Since  $\gamma^0$  satisfies the same equation in  $L^1$ , we have  $\gamma^0 - \gamma = \mathcal{L}_*(\gamma^0 - \gamma)$  hence  $\gamma^0 - \gamma = 0$  by the same argument as above [ $|\gamma^0 - \gamma|$  is in  $L^1$ , with "mass" in some  $V_\alpha$  because  $H(u_1)$  has measure 0, and this is sent to  $V_0$  by  $\mathcal{L}_*^n$  for some  $n$ ]. Thus  $\gamma^0$  is continuous of bounded variation on the intervals  $W_j$  for  $j = 1, \dots, N$ , and  $\phi^0$  has bounded variation on  $[a, b]$ , with possible discontinuities only at  $f^n u_1$  for  $n = 0, \dots, N$ , and  $\phi^0(a) = \phi^0(b) = 0$ . We have

$$\mathcal{L}_{(1)} \phi^0 - c_0^0 \psi_0 + \sum_{n=0}^{\infty} c_n^0 \chi_n = \phi^0$$

Therefore, hyperbolicity along the periodic orbit of  $u_1$  shows that  $\phi^0$  cannot have discontinuities, and this proves part (b) of Proposition 15.  $\square$

This also concludes the proof of Theorem 9.  $\square$

## 16 Remarks.

(a) Theorem 9 shows that the density  $\rho(x)$  of the unique a.c.i.m.  $\rho(x) dx$  for  $f$  can be written as the sum of spikes  $\approx |x - f^n b|^{-1/2} \theta_n(x)$  (where  $\theta_n$  vanishes unless  $x > f^n b$  or  $x < f^n b$ ) and a continuous background  $\phi(x)$ . In fact, one can also write  $\rho(x)$  as the sum of singular terms  $\approx |x - f^n b|^{-1/2} \theta_n(x)$ ,  $|x - f^n b|^{1/2} \theta_n(x)$  and a background  $\phi(x)$  which is now differentiable. This result is discussed in Appendix A. It seems clear that one could write  $\rho(x)$  as a sum of terms  $|x - f^n b|^{k/2} \theta_n(x)$  with  $k = -1, 1, \dots, \frac{2\ell-1}{2}$  and a background  $\phi(x)$  of class  $C^\ell$ , but we have not written a proof of this.

(b) Let  $u \in (-\infty, u_1) \cup (u_1, v_1) \cup (v_2, u_2) \cup (u_2, \infty)$  and choose  $w \in \{u_1, u_2, v_1, v_2\}$  such that  $w$  is an endpoint of the interval containing  $u$ . If  $\pm(w - u) > 0$  and  $\theta_\pm$  is the characteristic function of  $\{x : (w - x)(x - u) > 0\}$  we define

$$\psi_{(u\pm)}(x) = \frac{w - x}{w - u} \cdot |x - u|^{-1/2} \theta_\pm(x)$$

or a similar expression with the same singularity at  $u$ , greater smoothness at  $w$ , and/or  $\int \psi_{(u\pm)} = 0$ . [Note that the  $\psi_n$  are of this form]. Claim: if  $u \in \tilde{H}$ , there exists a

unique  $(\phi_\alpha) \in \mathcal{A}_1$  such that  $\phi_\alpha = \psi_{(u\pm)}|V_\alpha$  for all  $\alpha$ ; furthermore  $\|(\phi_\alpha)\|_1$  has a bound independent of  $u\pm$ . These results are proved in Appendix B (assuming  $\gamma < \alpha^{-1/2}$ ).

Note that if  $((\phi_\alpha), (c_n)) \in \mathcal{A}$  and  $c_0 = c_1 = 0$ , there is  $(\tilde{\phi}_\alpha) \in \mathcal{A}_1$  such that  $\Delta(\tilde{\phi}_\alpha) = w((\phi_\alpha), (c_n))$ . It seems thus that we might have replaced  $\mathcal{A}$  by  $\mathcal{A}_1$  in our earlier discussions. However, separating the spikes  $(c_n)$  from the background  $(\phi_\alpha)$  was needed in the spectral study of  $\mathcal{L}$ .

(c) The eigenvector  $\Phi^0$  of  $\mathcal{L}$  corresponding to the eigenvalue 1 (with  $w\Phi^0 \geq 0$ ,  $\int w\Phi^0 = 1$ ) depends continuously on  $f$ . To make sense of this statement we may consider a one-parameter family  $(f_\kappa)$  such that  $f_0 = f$ . We let  $H_\kappa, \tilde{H}_\kappa$  (hyperbolic sets) and  $\mathcal{A}_{1\kappa}$  (Banach space) reduce to  $H, \tilde{H}$  and  $\mathcal{A}_1$  when  $\kappa = 0$ . We restrict  $\kappa$  to a compact set  $K$  such that  $f_\kappa^3 c_\kappa \in \tilde{H}_\kappa$  (where  $c_\kappa$  is the critical point of  $f_\kappa$ ). The intervals  $V_{\kappa\alpha}$  associated with  $H_\kappa$  can be mapped to the  $V_\alpha$  associated with  $H$ , providing an identification  $\eta_\kappa : \mathcal{A}_{\kappa 1} \rightarrow \mathcal{A}_1$ . There are natural definitions of  $\mathcal{L}_\kappa : \mathcal{A}_{\kappa 1} \oplus \mathcal{A}_2 \rightarrow \mathcal{A}_{\kappa 1} \oplus \mathcal{A}_2$  and the eigenvector  $\Phi_\kappa^0$  reducing to  $\mathcal{L}$  and  $\Phi^0$  when  $\kappa = 0$ . We claim that  $\kappa \mapsto \Phi_\kappa^\times = (\eta_\kappa, \mathbf{1})\Phi_\kappa^0$  is a continuous function  $K \rightarrow \mathcal{A}_1 \oplus \mathcal{A}_2$ . This result is proved in Appendix C. It implies that, if  $A$  is smooth,  $\kappa \rightarrow \langle \Phi_{f_\kappa}^0, A \rangle$  is continuous on  $K$ . The weight of the  $n$ -th spike is  $C_0 \prod_{k=1}^n |f'_\kappa(f_\kappa^{k-1}b_\kappa)|^{-1/2}$  and its speed is

$$\frac{d}{d\kappa} f_\kappa^n b_\kappa = \prod_{k=1}^n f'_\kappa(f_\kappa^{k-1}b_\kappa) \frac{db_\kappa}{d\kappa} + \sum_{\ell=1}^n \prod_{k=\ell+1}^n f'_\kappa(f_\kappa^{k-1}b_\kappa) f_\kappa^*(f_\kappa^{\ell-1}b_\kappa) \quad \text{with} \quad f_\kappa^* = \frac{df_\kappa}{d\kappa}$$

The weight may be roughly estimated as  $\sim \alpha^{n/2}$  and the speed as  $\sim \alpha^{-n}$  for some  $\alpha \in (0, 1)$ , suggesting that  $\kappa \rightarrow \langle \Phi_{f_\kappa}^0, A \rangle$  is  $\frac{1}{2}$ -Hölder on  $K$ .

## 17 Informal study of the differentiability of $f \mapsto \langle \Phi_f^0, A \rangle$ .

Writing  $\Phi_f^0$  instead of  $\Phi^0$  we want to study the change of  $\langle \Phi_f^0, A \rangle = \int dx (w\Phi_f^0)(x)A(x)$  when  $f$  is replaced by  $\hat{f}$  close to  $f$  (and the critical orbit  $\hat{f}^k \hat{c}$  for  $k \geq 3$  is in the perturbed hyperbolic set  $\hat{\tilde{H}}$ ). Writing  $g = \text{id} - \hat{f}(\hat{c}) + f(c)$ , we see that  $\hat{f}$  is conjugate to  $g \circ \hat{f} \circ g^{-1}$ , which has maximum  $f(c)$  at  $g(\hat{c})$ . With proper choice of the inverse  $f^{-1}$  we have  $f^{-1} \circ (g \circ \hat{f} \circ g^{-1}) = h$  close to  $\text{id}$ , hence  $g \circ \hat{f} \circ g^{-1} = f \circ h$  and  $(h \circ g) \circ \hat{f} \circ (h \circ g)^{-1} = h \circ f$ , *i.e.*,  $\hat{f}$  is conjugate to  $h \circ f$  and we may write

$$\langle \Phi_{\hat{f}}^0, A \rangle = \langle \Phi_{h \circ f}^0, A \circ h \circ g \rangle$$

The differentiability of  $\hat{f} \mapsto A \circ h \circ g$  is trivial, and we concentrate on the study of  $h \mapsto \langle \Phi_{h \circ f}^0, A \rangle$ . Writing  $h = \text{id} + X$ , where  $X$  is analytic, we see that the change  $\delta(w\Phi_f^0)$  when  $f$  is replaced by  $(\text{id} + X) \circ f$  is, to first order in  $X$ , formally

$$(1 - \mathcal{L})^{-1} \mathcal{D}(-X\Phi_f^0)$$

where  $\mathcal{D}$  denotes differentiation. [The above formula is standard first order perturbation calculation, and we have omitted the  $w$  map from our formula].

Writing  $\Phi_f^0 = ((\phi_\alpha^0), (C_n))$ , we can identify  $\mathcal{D}(-X((\phi_\alpha^0), 0))$  with an element  $\Phi^\times$  of  $\mathcal{A}$  (so that  $w\Phi^\times = \mathcal{D}(Xw((\phi_\alpha^0), 0))$  and  $\int dx w\Phi^\times(x) = 0$ , use Appendix A) which is easy to study, and we are left to analyze the singular part  $\mathcal{D}(-X(0, (C_n)))$ . To study this singular part we shall write  $(0, (C_n)) = \sum_{n=0}^{\infty} C_n \psi_{(f^n b)}$ , and use the equivalence  $\sim$  modulo the elements of  $\mathcal{A}$ . We extend the domain of definition of  $\mathcal{L}$  so that  $\mathcal{L}\psi_{(u)} \sim |f'(u)|^{-1/2} \psi_{(fu)}$ , where we use the notation  $\psi_{(u\pm)}$  of Section 16(b), but omit the  $\pm$ , and we assume that  $\int \psi_{(u)} = 0$ . We have thus

$$\begin{aligned} \mathcal{D}(-X(0, (C_n))) &\sim - \sum_{n=0}^{\infty} C_n X(f^n b) \mathcal{D}\psi_{(f^n b)} \sim \sum_{n=0}^{\infty} C_n X(f^n b) \frac{d}{du} \psi_{(u)} \Big|_{u=f^n b} \\ &= \sum_{n=0}^{\infty} C_n X(f^n b) \left[ \prod_{k=0}^{n-1} f'(f^k b) \right]^{-1} \frac{d}{db} \psi_{(f^n b)} \sim \sum_{n=0}^{\infty} X(f^n b) \left[ \prod_{k=0}^{n-1} f'(f^k b) \right]^{-1} \frac{d}{db} \mathcal{L}^n C_0 \psi_{(b)} \end{aligned}$$

We may thus write (introducing  $(1 - \lambda\mathcal{L})^{-1}$  instead of  $(1 - \mathcal{L})^{-1}$ )

$$\begin{aligned} (1 - \lambda\mathcal{L})^{-1} \mathcal{D}(-X(0, (C_n))) &\sim \sum_{n=0}^{\infty} X(f^n b) \left[ \prod_{k=0}^{n-1} f'(f^k b) \right]^{-1} \lambda^{-n} \frac{d}{db} (1 - \lambda\mathcal{L})^{-1} (\lambda\mathcal{L})^n C_0 \psi_{(b)} \\ &= \sum_{n=0}^{\infty} X(f^n b) \left[ \prod_{k=0}^{n-1} f'(f^k b) \right]^{-1} \lambda^{-n} \frac{d}{db} (1 - \lambda\mathcal{L})^{-1} C_0 \psi_{(b)} - Z \end{aligned}$$

where

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} X(f^n b) \left[ \prod_{k=0}^{n-1} f'(f^k b) \right]^{-1} \lambda^{-n} \frac{d}{db} \sum_{\ell=0}^{n-1} (\lambda\mathcal{L})^\ell C_0 \psi_{(b)} \\ &\sim \sum_{n=0}^{\infty} X(f^n b) \left[ \prod_{k=0}^{n-1} f'(f^k b) \right]^{-1} \sum_{\ell=0}^{n-1} \lambda^{-n+\ell} \left| \prod_{k=0}^{\ell-1} f'(f^k b) \right|^{-1/2} \frac{d}{db} C_0 \psi_{(f^\ell b)} \\ &= \sum_{n=0}^{\infty} X(f^n b) \sum_{\ell=0}^{n-1} \lambda^{-n+\ell} \left[ \prod_{k=\ell}^{n-1} f'(f^k b) \right]^{-1} \left| \prod_{k=0}^{\ell-1} f'(f^k b) \right|^{-1/2} \frac{d}{du} C_0 \psi_{(u)} \Big|_{u=f^\ell b} \\ &\sim -\mathcal{D} \sum_{n=0}^{\infty} X(f^n b) \sum_{\ell=0}^{n-1} \lambda^{-n+\ell} \left[ \prod_{k=\ell}^{n-1} f'(f^k b) \right]^{-1} C_\ell \psi_\ell \\ &= -\mathcal{D} \sum_{r=1}^{\infty} \sum_{\ell=0}^{\infty} X(f^{\ell+r} b) \lambda^{-r} \left[ \prod_{k=0}^{r-1} f'(f^{\ell+k} b) \right]^{-1} C_\ell \psi_\ell \\ &= -\mathcal{D} \sum_{\ell=0}^{\infty} C_\ell \psi_\ell \sum_{r=1}^{\infty} \lambda^{-r} \left[ \prod_{k=0}^{r-1} f'(f^{\ell+k} b) \right]^{-1} X(f^{\ell+r} b) \end{aligned}$$

We have thus an (informal) proof of the following result

For  $\ell = 0, 1, \dots$ , define

$$F_\ell(X) = \sum_{n=1}^{\infty} \lambda^{-n} \left[ \prod_{k=0}^{n-1} f'(f^{k+\ell}b) \right]^{-1} X(f^{n+\ell}b)$$

which are holomorphic functions of  $\lambda$  when  $|\lambda| > \alpha$ . Then the susceptibility function

$$\Psi(\lambda) = \langle (1 - \lambda \mathcal{L})^{-1} \mathcal{D}(-X \Phi_f^0), A \rangle$$

has the form

$$\Psi(\lambda) \sim (X(b) + F_0(X)) \frac{d}{db} \langle (1 - \lambda \mathcal{L})^{-1} C_0 \psi_{(b)}, A \rangle - \sum_{\ell=0}^{\infty} F_\ell(X) C_\ell \langle \psi_\ell, \mathcal{D}A \rangle$$

The derivative  $\frac{d}{db} \langle (1 - \lambda \mathcal{L})^{-1} C_0 \psi_{(b)}, A \rangle$  exists as a distribution, but is in principle a divergent quantity for given  $b$ . The corresponding term disappears however if  $X(b) + F_0(X) = 0$ , and we are then left with a finite expression, meromorphic in  $\lambda$  for  $\alpha < |\lambda| < \min(\beta^{-1}, \alpha^{-1/2})$  and holomorphic when  $\alpha < |\lambda| \leq 1$ .

Note that in writing the equivalence  $\sim$  we have omitted terms with the singularities of  $(1 - \lambda \mathcal{L})^{-1}$ ; this explains the meromorphic contributions for  $|\lambda| > 1$ . The condition  $X(b) + F_0(X) = 0$  for  $\lambda = 1$  is known as *horizontality* (see the discussion in Section 19 below).

## 18 A modified susceptibility function $\Psi(X, \lambda)$ .

At this point we extend the definition of the operator  $\mathcal{L}$  to  $\mathcal{L}^\sim$  acting on a larger space. Remember that  $\mathcal{L}$  was obtained from the transfer operator  $\mathcal{L}_{(1)}$  by separating the spikes  $\psi_n$  from the background in order to obtain better spectral properties. We now also introduce derivatives  $\psi'_n$  of spikes, so that the transfer operator sends  $\psi'_n$  to

$$\frac{f'(f^n b)}{|f'(f^n b)|^{1/2}} \psi'_{n+1} + \text{a term in } w(\mathcal{A}_1 + \mathcal{A}_2)$$

The coefficients of  $\psi'_n$  form an element of  $\mathcal{A}_3 = \{(Y_n) : \| (Y_n) \|_3 = \sup_n \delta^n |Y_n| < \infty\}$ . We define  $\mathcal{L}^\sim$  on  $\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3$  so that

$$\mathcal{L}^\sim = \begin{pmatrix} \mathcal{L}_0 + \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_5 \\ \mathcal{L}_3 & \mathcal{L}_4 & \mathcal{L}_6 \\ 0 & 0 & \mathcal{L}_7 \end{pmatrix}$$

where we omit the explicit definition of  $\mathcal{L}_5, \mathcal{L}_6$ , and let

$$\mathcal{L}_7 \left( \frac{Z_n}{\prod_{k=0}^{n-1} |f'(f^k b)|^{1/2}} \right) = \left( \frac{\tilde{Z}_n}{\prod_{k=0}^{n-1} |f'(f^k b)|^{1/2}} \right)$$

with  $\tilde{Z}_0 = 0$ ,  $\tilde{Z}_n = f'(f^{n-1}b)Z_{n-1}$  for  $n > 0$ . Since

$$\begin{pmatrix} 0 & 0 & \mathcal{L}_5 \\ 0 & 0 & \mathcal{L}_6 \\ 0 & 0 & \mathcal{L}_7 \end{pmatrix} \mathcal{L} = 0$$

we have

$$\mathcal{L}^{\sim n} = \mathcal{L}^n + \sum_{k=1}^n \mathcal{L}^{k-1}(\mathcal{L}_5 + \mathcal{L}_6)\mathcal{L}_7^{n-k} + \mathcal{L}_7^n$$

and formally

$$(\mathbf{1} - \lambda \mathcal{L}^{\sim})^{-1} = (\mathbf{1}_{12} - \lambda \mathcal{L})^{-1} + (\mathbf{1}_3 - \lambda \mathcal{L}_7)^{-1} + (\mathbf{1}_{12} - \lambda \mathcal{L})^{-1} \lambda (\mathcal{L}_5 + \mathcal{L}_6) (\mathbf{1}_3 - \lambda \mathcal{L}_7)^{-1}$$

where  $\mathbf{1}_{12}$  and  $\mathbf{1}_3$  denote the identity on  $\mathcal{A}_1 \oplus \mathcal{A}_2$  and  $\mathcal{A}_3$  respectively.

For  $\lambda$  close to 1,  $(\mathbf{1}_3 - \lambda \mathcal{L}_7)^{-1}$  and thus  $(\mathbf{1} - \lambda \mathcal{L}^{\sim})^{-1}$  are not well defined. But there is a natural definition of a left inverse  $\mathcal{L}_{7L}^{-1}$  of  $\mathcal{L}_7$  where

$$\mathcal{L}_{7L}^{-1} \left( \frac{Z_n}{\prod_{k=0}^{n-1} |f'(f^k b)|^{1/2}} \right) = \left( \frac{\tilde{Z}_n}{\prod_{k=0}^{n-1} |f'(f^k b)|^{1/2}} \right)$$

with  $\tilde{Z}_n = f'(f^n b)^{-1} Z_{n+1}$  for  $n \geq 0$ . The spectral radius of  $\mathcal{L}_{7L}^{-1}$  is thus  $\leq \alpha^{1/2}/\delta$ . This gives natural left inverses

$$(\mathbf{1}_3 - \lambda \mathcal{L}_7)_L^{-1} = - \sum_{n=1}^{\infty} \lambda^{-n} \mathcal{L}_{7L}^{-n}$$

for  $|\lambda| > \alpha^{1/2}/\delta$ , and

$$(\mathbf{1} - \lambda \mathcal{L}^{\sim})_L^{-1} = (\mathbf{1}_{12} - \lambda \mathcal{L})^{-1} + (\mathbf{1}_3 - \lambda \mathcal{L}_7)_L^{-1} + (\mathbf{1}_{12} - \lambda \mathcal{L})^{-1} \lambda (\mathcal{L}_5 + \mathcal{L}_6) (\mathbf{1}_3 - \lambda \mathcal{L}_7)_L^{-1}$$

when  $|\lambda| > \alpha^{1/2}/\delta$  and  $(\mathbf{1}_{12} - \lambda \mathcal{L})^{-1}$  exists. This gives a modified susceptibility function

$$\Psi_L(\lambda) = \langle (\mathbf{1} - \lambda \mathcal{L}^{\sim})_L^{-1} \mathcal{D}(-X \Phi_f^0), A \rangle$$

meromorphic in  $\lambda$  for  $\alpha < |\lambda| < \min(\beta^{-1}, \alpha^{-1/2})$  and holomorphic for  $\alpha < |\lambda| \leq 1$ .

Note that the  $\mathcal{A}_3$  part of  $\mathcal{D}(-X \Phi_f^0)$  is

$$(Y_n) = \left( \frac{-X(f^n b)}{\frac{1}{2} |f''(c)|^{1/2} \prod_{k=0}^{n-1} |f'(f^k b)|^{1/2}} \right)_{n \geq 0}$$

where  $\sup_n |X(f^n b)| < \infty$ . Therefore, for small  $|\lambda|$ ,

$$(\mathbf{1}_3 - \lambda \mathcal{L}_7)^{-1} (Y_n) = \left( \frac{-\sum_{k=0}^n \lambda^k (\prod_{\ell=1}^k f'(f^{n-\ell} b)) X(f^{n-k} b)}{\frac{1}{2} |f''(c)|^{1/2} \prod_{k=0}^{n-1} |f'(f^k b)|^{1/2}} \right)_{n \geq 0}$$

because the right-hand side is in  $\mathcal{A}_3$ . Note that the right-hand side is also in  $\mathcal{A}_3$  under the condition

$$\sum_{n=0}^{\infty} \lambda^{-n} \left( \prod_{k=0}^{n-1} f'(f^k b) \right)^{-1} X(f^n b) = 0 \quad (*)$$

because this condition implies

$$-\sum_{k=0}^n \lambda^{-k} \left( \prod_{\ell=0}^{k-1} f'(f^\ell b) \right)^{-1} X(f^k b) = \sum_{k=n+1}^{\infty} \lambda^{-k} \left( \prod_{\ell=0}^{k-1} f'(f^\ell b) \right)^{-1} X(f^k b)$$

hence, multiplying by  $\lambda^n \prod_{\ell=0}^{n-1} f'(f^\ell b)$ ,

$$-\sum_{k=0}^n \lambda^{n-k} \left( \prod_{\ell=k}^{n-1} f'(f^\ell b) \right) X(f^k b) = \sum_{k=n+1}^{\infty} \lambda^{n-k} \left( \prod_{\ell=n}^{k-1} f'(f^\ell b) \right)^{-1} X(f^k b)$$

or

$$-\sum_{k=0}^n \lambda^k \left( \prod_{\ell=1}^k f'(f^{n-\ell} b) \right) X(f^{n-k} b) = \sum_{k=1}^{\infty} \lambda^{-k} \left( \prod_{\ell=0}^{k-1} f'(f^{n+\ell} b) \right)^{-1} X(f^{n+k} b)$$

for each  $n$ , provided  $|\lambda| > \alpha$ . We have proved that:

*Under the condition (\*), a resummation of the series defining*

$$\langle (\mathbf{1} - \lambda \mathcal{L}^\sim)^{-1} \mathcal{D}(-X \Phi_f^0), A \rangle$$

*yields  $\Psi_L(\lambda)$ .*

It is then natural to define a modified susceptibility function  $\Psi(X, \lambda)$  by

$$(X, \lambda) \mapsto \Psi(X, \lambda) = \Psi_L(\lambda) \quad \text{on} \quad \{(X, \lambda) : (*) \text{ holds}\}$$

Note that the left-hand side of (\*) is equal to the quantity  $X(b) + F_0(X)$  met in Section 17, and that (\*) with  $\lambda = 1$  reduces to the horizontality condition.

We conclude with a rigorous result agreeing in part with the informal study in Section 17, in part with a conjecture of Baladi [3], Baladi and Smania [5].

**19 Theorem** (differentiability along topological conjugacy classes).

*Let  $f_\kappa = h_\kappa \circ f$  where the  $h_\kappa$  are real analytic, depend smoothly on  $\kappa$ , and  $f_\kappa^3 c = \xi_\kappa f^3 c$  identically in  $\kappa$ . [This last condition expresses that  $f_\kappa$  belongs to a conjugacy class, and  $\xi_k : H \rightarrow H_\kappa$  is the conjugacy defined in Appendix C]. Then, if  $A$  is smooth,  $\kappa \mapsto \langle \Phi_\kappa^0, A \rangle = \int dx (w \Phi_{f_\kappa}^0)(x) A(x)$  is continuously differentiable. Furthermore*

$$\left. \frac{d}{d\kappa} \langle \Phi_{f_\kappa}^0, A \rangle \right|_{\kappa=0} = \Psi(X, 1)$$



where  $\Psi(X, \lambda)$  is defined in Section 18 with  $X = \frac{d}{d\kappa} h_\kappa|_{\kappa=0}$ , and  $\Psi(X, \lambda)$  is holomorphic for  $\alpha < |\lambda| \leq 1$ , meromorphic for  $\alpha < |\lambda| < \min(\beta^{-1}, \alpha^{-1/2})$ .

[The value  $\kappa = 0$  plays no special role, and is chosen for notational simplicity in the formulation of the theorem].

Our notion of topological conjugacy class is a special case of that discussed in [1].

Note that  $\xi_0 = \text{id}$ , and that  $\xi_\kappa$  depends differentiably on  $\kappa$ . Since  $f_\kappa^3 c = \xi_\kappa f^3 c$  and  $f_\kappa \xi_\kappa = \xi_\kappa f$  on  $H$ , we have  $f_\kappa^n c = \xi_\kappa f^n c$  for  $n \geq 3$  and by differentiation (writing  $\xi' = \frac{d}{d\kappa} \xi_\kappa|_{\kappa=0}$ ):

$$\sum_{k=1}^n [\prod_{\ell=k}^{n-1} f'(f^\ell c)] X(f^k c) = \xi'(f^n c)$$

or

$$\sum_{k=1}^n [\prod_{\ell=1}^{k-1} f'(f^\ell c)]^{-1} X(f^k c) = [\prod_{\ell=1}^{n-1} f'(f^\ell c)]^{-1} \xi'(f^n c)$$

and letting  $n \rightarrow \infty$ :

$$\sum_{k=1}^{\infty} [\prod_{\ell=1}^{k-1} f'(f^\ell c)]^{-1} X(f^k c) = 0 \quad \text{or} \quad \sum_{n=0}^{\infty} [\prod_{k=0}^{n-1} f'(f^k b)]^{-1} X(f^n b) = 0$$

This is the horizontality condition derived much more generally in [1].

The proof of the theorem will be based on Appendices A, B, C, and use particularly the notation of Appendix C. We write  $\Phi_{f_\kappa}^0 = \Phi_\kappa^0$  and recall that the expression

$$\langle \Phi_\kappa^0, A \rangle_\kappa = \int dx (w_\kappa \Phi_\kappa^0)(x) A(x) = \sum_\alpha \int_{V_{\kappa\alpha}} \phi_{\kappa\alpha}^0 A(x) dx + \sum_n c_{\kappa n}^0 \int \psi_{\kappa n}(x) A(x) dx$$

depends explicitly on the intervals  $V_{\kappa\alpha}$  and the points  $f_\kappa^k c$  for  $k \geq 1$ . We shall first prove the existence of  $\frac{d}{d\kappa} \langle \Phi_\kappa^0, A \rangle_\kappa|_{\kappa=0} = \lim_{\kappa \rightarrow 0} \frac{1}{\kappa} \int (w_\kappa \Phi_\kappa^0 - w \Phi^0) A$  and give an expression involving only the intervals  $V_\alpha$  and the points  $f^k c$  (corresponding to  $\kappa = 0$ ). Then we shall transform the expression obtained to the form  $\Psi(X, 1)$ .

Since the map  $\xi_\kappa : H \rightarrow H_\kappa$  depends smoothly on  $\kappa$  (in particular  $f'_\kappa(f_\kappa^k b_\kappa) = f'_\kappa(\xi_\kappa f^k b)$  is continuous uniformly in  $k$ ), it is easily seen that the operator  $\mathcal{L}_\kappa^\times$  defined in Appendix C now depends continuously and even differentiably on  $\kappa$ .

We may write

$$\begin{aligned} \langle \Phi_\kappa^0, A \rangle_\kappa &= \sum_\alpha \int_{V_{\kappa\alpha}} \phi_{\kappa\alpha}^0(x) A(x) dx + \sum_n c_{\kappa n}^0 \int \psi_{\kappa n}(x) A(x) dx \\ &= \langle ((\phi_{\kappa\alpha}^0), (c_{\kappa n}^0)), ((A|V_{\kappa\alpha}), A) \rangle_\kappa \\ &= \langle \Phi_\kappa^0, ((A|V_{\kappa\alpha}), 0) \rangle_\kappa + \langle \Phi_\kappa^0, (0, (c_{\kappa n}^0)) \rangle_\kappa \end{aligned}$$

For notational simplicity we study the derivative of this quantity at  $\kappa = 0$  but the proof will show that the derivative depends continuously on  $\kappa$ . We have

$$\frac{1}{\kappa} \left[ \langle \Phi_\kappa^0, A \rangle_\kappa - \langle \Phi^0, A \rangle \right] = I + II$$

where

$$\begin{aligned} II &= \frac{1}{\kappa} \sum_n \int [c_{\kappa n}^0 \psi_{\kappa n}(x) - c_n^0 \psi_n(x)] A(x) dx \\ &\rightarrow \sum_n \int \left[ \frac{dc_{\kappa n}^0}{d\kappa} \psi_n(x) + c_n^0 \frac{d}{d\kappa} \psi_{\kappa n}(x) \right] A(x) dx \Big|_{\kappa=0} \end{aligned}$$

$[\frac{d}{d\kappa} \psi_{\kappa n}]$  is a distribution with singular part  $\frac{d}{d\kappa} |x - f_\kappa^n b_\kappa|^{-1/2}$ ; integrating by part over  $x$ , and using  $f_\kappa^n b_\kappa = \xi_\kappa f^n b$  for  $k \geq 2$ , we see that the right-hand side makes sense, and is the limit of the left-hand side when  $\kappa \rightarrow 0$ ].

We also have

$$\langle \Phi_\kappa^0, ((A|V_{\kappa\alpha}), 0) \rangle_\kappa = \langle \Phi_\kappa^\times, ((A_{\kappa\alpha}), 0) \rangle$$

where  $A_{\kappa\alpha} = (A|V_{\kappa\alpha}) \circ \tilde{\eta}_{\kappa\alpha}^{-1}$ , so that

$$I = \left\langle \frac{\Phi_\kappa^\times - \Phi_0^\times}{\kappa}, ((A_{\kappa\alpha}), 0) \right\rangle + \left\langle \Phi_0^\times, \left( \frac{A_{\kappa\alpha} - A_{0\alpha}}{\kappa}, 0 \right) \right\rangle$$

and the second term is readily seen to tend to a limit when  $\kappa \rightarrow 0$ . In the first term remember that for  $\kappa = 0$  we have  $\Phi_\kappa^\times = \Phi_0^\times = \Phi^0$ , and  $\mathcal{L}_\kappa^\times = \mathcal{L}_0^\times = \mathcal{L}$ . Also

$$(\mathbf{1} - \mathcal{L})(\Phi_\kappa^\times - \Phi_0^\times) = (\mathcal{L}_\kappa^\times - \mathcal{L}_0^\times) \Phi_\kappa^\times$$

hence

$$\Phi_\kappa^\times - \Phi_0^\times = (\mathbf{1} - \mathcal{L})^{-1} (\mathcal{L}_\kappa^\times - \mathcal{L}_0^\times) \Phi_\kappa^\times$$

Since  $(\mathbf{1} - \mathcal{L})^{-1}$  is bounded and  $\kappa \mapsto \mathcal{L}_\kappa^\times$  differentiable, we have

$$\left\langle \frac{\Phi_\kappa^\times - \Phi_0^\times}{\kappa}, ((A_{\kappa\alpha}), 0) \right\rangle \rightarrow \left\langle (\mathbf{1} - \mathcal{L})^{-1} \left( \frac{d}{d\kappa} \mathcal{L}_\kappa^\times \Big|_{\kappa=0} \right) \Phi^0, ((A_{0\alpha}), 0) \right\rangle$$

when  $\kappa \rightarrow 0$ , proving that  $\kappa \mapsto \langle \Phi_\kappa^0, A \rangle$  is differentiable.

If we replace in the above calculation the Banach space  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$  by  $\mathcal{A}' = \mathcal{A}'_1 \oplus \mathcal{A}'_2$  as in Appendix A, we obtain an expression of  $\frac{d}{d\kappa} \langle \Phi_\kappa^0, A \rangle_\kappa \Big|_{\kappa=0}$  that can be re-expressed in terms of the  $\psi'_n$ ,  $\psi_n$  and an element of  $\mathcal{A}_1$ . We may thus write

$$\frac{d}{d\kappa} \langle \Phi_\kappa^0, A \rangle_\kappa \Big|_{\kappa=0} = \langle \tilde{\Phi}, A \rangle^\sim$$

where  $\tilde{\Phi} \in \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$ . The part  $\tilde{\Phi}_3$  of  $\tilde{\Phi}$  in  $\mathcal{A}_3$  is uniquely determined by  $A \mapsto \langle \tilde{\Phi}, A \rangle^\sim$ ; the calculation of II above shows that  $n$ -th component of  $\tilde{\Phi}_3$  is

$$-\frac{d}{d\kappa} f_\kappa^n b_\kappa \Big|_{\kappa=0} c_n^0 = -\frac{d}{d\kappa} f_\kappa^{n+1} c \Big|_{\kappa=0} c_n^0$$

$$= - \sum_{k=1}^{n+1} X(f^k c) \left( \prod_{\ell=k}^n f'(f^\ell c) \right) c_n^0 = - \sum_{k=0}^n X(f^k b) \left( \prod_{\ell=k}^{n-1} f'(f^\ell b) \right) c_n^0$$

and as a result

$$(\mathbf{1} - \mathcal{L}_7) \tilde{\Phi}_3 = (-X(f^n b) C_n^0)_{n \geq 0}$$

$$\tilde{\Phi}_3 = (\mathbf{1} - \mathcal{L}_7)_L^{-1} (-X(f^n b) C_n^0)_{n \geq 0}$$

The part  $\Phi^*$  of  $\tilde{\Phi}$  in  $\mathcal{A}_1 \oplus \mathcal{A}_2$  is not uniquely determined (because of the ambiguity discussed in Appendix B); this part satisfies  $\int w \Phi^* = 0$ .

If  $\mathcal{L}_{(1)\kappa}$  is the transfer operator corresponding to  $f_\kappa$ , we have  $\mathcal{L}_{(1)\kappa} w_\kappa \Phi_\kappa^0 = w_\kappa \Phi_\kappa^0$ , hence

$$(\mathbf{1} - \mathcal{L}_{(1)})(w_\kappa \Phi_\kappa^0 - w \Phi^0) = (\mathcal{L}_{(1)\kappa} - \mathcal{L}_{(1)}) w_\kappa \Phi_\kappa^0$$

Therefore (using the fact that we may let  $\mathcal{L}_{(1)}$  act on  $A$ ) we have

$$\begin{aligned} \langle (\mathbf{1} - \mathcal{L}^\sim) \tilde{\Phi}, A \rangle^\sim &= \lim_{\kappa \rightarrow 0} \int A \frac{1}{\kappa} (\mathbf{1} - \mathcal{L}_{(1)})(w_\kappa \Phi_\kappa^0 - w \Phi^0) \\ &= \lim_{\kappa \rightarrow 0} \int A \frac{1}{\kappa} (\mathcal{L}_{(1)\kappa} - \mathcal{L}_{(1)}) w_\kappa \Phi_\kappa^0 = \lim_{\kappa \rightarrow 0} \int A \frac{1}{\kappa} (\text{id}^* - h_{-\kappa}^*) w_\kappa \Phi_\kappa^0 = \lim_{\kappa \rightarrow 0} \int A \frac{1}{\kappa} (h_\kappa^* - \text{id}^*) w \Phi^0 \end{aligned}$$

where  $h^*$  denotes the direct image of a measure (here a  $L^1$  function) under  $h$ , and the last equality uses the existence of a continuous derivative for  $\kappa \mapsto \langle \Phi_\kappa^0, A \rangle_\kappa$ . According to Appendix A we may write  $w \Phi^0$  as a sum of terms  $C_n^{(0)} \psi_n^{(0)}$ ,  $C_n^{(1)} \psi_n^{(1)}$ , and a differentiable background. Corresponding to this we may identify  $\lim_{\kappa \rightarrow 0} \frac{1}{\kappa} (h_\kappa^* - \text{id}^*) \Phi^0$  with a naturally defined element  $\mathcal{D}(-X \Phi^0)$  of  $\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$ , where  $\mathcal{D}$  denotes differentiation. We write  $\mathcal{D}(-X \Phi^0) = (D^*, D_3)$  with  $D^* \in \mathcal{A}_1 \oplus \mathcal{A}_2$ ,  $D_3 \in \mathcal{A}_3$ . Since the coefficient of  $\psi_n'$  in  $\mathcal{D}(-X \Phi^0)$  is  $-X(f^n b) c_n^0$ , we have  $D_3 = (\mathbf{1} - \mathcal{L}_7) \tilde{\Phi}_3$ . With  $\tilde{\Phi} = (\Phi^*, \tilde{\Phi}_3)$  we have thus

$$\langle (\mathbf{1} - \mathcal{L}^\sim)(\Phi^*, \tilde{\Phi}_3), A \rangle^\sim = \langle \mathcal{D}(-X \Phi^0), A \rangle^\sim$$

and

$$\langle (\mathbf{1} - \mathcal{L}) \Phi^*, A \rangle = \langle \mathcal{D}(-X \Phi^0) - (\mathbf{1} - \mathcal{L}^\sim)(0, \tilde{\Phi}_3), A \rangle$$

In particular  $\int w [\mathcal{D}(-X \Phi^0) - (\mathbf{1} - \mathcal{L}^\sim)(0, \tilde{\Phi}_3)] = 0$  and we may define

$$\Phi = (\mathbf{1} - \mathcal{L})^{-1} [\mathcal{D}(-X \Phi^0) - (\mathbf{1} - \mathcal{L}^\sim)(0, \tilde{\Phi}_3)] \in \mathcal{A}$$

We have then  $\langle (\mathbf{1} - \mathcal{L})(\Phi^* - \Phi), A \rangle = 0$ , hence

$$w(\mathbf{1} - \mathcal{L})(\Phi^* - \Phi) = 0$$

hence

$$w(\Phi^* - \Phi) = \mathcal{L}_{(1)} w(\Phi^* - \Phi)$$

with  $\int w(\Phi^* - \Phi) = 0$ , so that  $w(\Phi^* - \Phi) = 0$ , and

$$\begin{aligned}\langle \Phi^*, A \rangle &= \langle \Phi, A \rangle = \langle (\mathbf{1} - \mathcal{L})^{-1}[\mathcal{D}(-X\Phi^0) - (\mathbf{1} - \mathcal{L}^\sim)(0, \tilde{\Phi}_3)], A \rangle \\ &= \langle (\mathbf{1} - \mathcal{L})^{-1}[D^* + \mathcal{L}_5 + \mathcal{L}_6]\tilde{\Phi}_3, A \rangle = \langle (\mathbf{1} - \mathcal{L})^{-1}[D^* + \mathcal{L}_5 + \mathcal{L}_6](\mathbf{1} - \mathcal{L}_7)_L^{-1}D_3, A \rangle \\ &= \langle (\mathbf{1} - \mathcal{L}^\sim)_L^{-1}(D^*, D_3), A \rangle - \langle (\mathbf{1} - \mathcal{L}_7)_L^{-1}D_3, A \rangle = \Psi(X, 1) - \langle (0, \tilde{\Phi}_3), A \rangle^\sim\end{aligned}$$

so that finally

$$\frac{d}{d\kappa} \langle \Phi_\kappa^0, A \rangle_{\kappa=0} = \langle \tilde{\Phi}, A \rangle^\sim = \Psi(X, 1)$$

as announced.  $\square$

Note that in [5], Baladi and Smania study (in the case of piecewise expanding maps) the more difficult problem of differentiability in horizontal directions (*i.e.*, directions tangent to a topological class). It appears likely that this could be done here also (as conjectured in [5]), but we have not tried to do so.

## 20 Discussion.

The codimension 1 condition  $X(b) + F_0(X) = 0$  for  $\lambda = 1$  expresses that  $X$  is a *horizontal* perturbation, which means that it is tangent to a topological class of unimodal maps (see [1] and references given there). In our case, a family  $(f_\kappa)$  is in a topological conjugacy class if  $f_\kappa^3 c_\kappa = \xi_\kappa f^3 c$  in the notation of Appendix C. The informal result obtained in Section 17 and the formal proof of differentiability along a topological conjugacy class given by Theorem 19 support the conjecture by Baladi and Smania [5] that the map  $f \mapsto \langle \Phi_f^0, A \rangle$  is differentiable (in the sense of Whitney) in horizontal directions, *i.e.*, along a curve tangent to a topological conjugacy class. Our theorem 19 also relates the derivative along a topological conjugacy class to a naturally defined susceptibility function. It seems unlikely that a derivative (in the sense of Whitney) exists in nonhorizontal directions. Note however that if  $f \mapsto \langle \Phi_f^0, A \rangle$  is nondifferentiable, it will be in a mild way: the "nondifferentiable" contribution to  $\Psi(\lambda)$  is, as we saw above, proportional to

$$\frac{d}{db} \langle (1 - \lambda \mathcal{L})^{-1} \psi_{(b)}, A \rangle \sim \sum_n \lambda^n \frac{d}{db} \langle \mathcal{L}^n \psi_{(b)}, A \rangle$$

where  $\langle \mathcal{L}^n \psi_{(b)}, A \rangle$  decreases exponentially with  $n$ , while  $\frac{d}{db} \langle \mathcal{L}^n \psi_{(b)}, A \rangle$  increases exponentially. Therefore, if one does not see the small scale fluctuations of  $b \mapsto \langle (1 - \lambda \mathcal{L})^{-1} \psi_{(b)}, A \rangle$ , this function will seem differentiable. But the singularities with respect to  $\lambda$  (with  $|\lambda| < 1$ ) may remain visible. In conclusion, a physicist may see singularities with respect to  $\lambda$  of a derivative (with respect to  $f$  or  $b$ ) while this derivative may not exist for a mathematician.

## A Appendix (proof of Remark 16(a)).

We return to the analysis in Section 10, and note that by an analytic change of variable  $x \mapsto \xi(x)$  we can get  $y = fx = b - \xi^2$  [we have indeed  $b - y = A(x - c)^2(1 + \beta(x) \cdot (x - c))$  with  $\beta$  analytic, and we can take  $\xi = (x - c)A^{1/2}(1 + \beta(x) \cdot (x - c))^{\frac{1}{2}}$ ]. Write  $\rho(x) dx = \tilde{\rho}(\xi) d\xi$  (where  $\tilde{\rho}$  is analytic near 0). The density of the image  $\delta(y) dy$  by  $f$  of  $\rho(x) dx = \tilde{\rho}(\xi) d\xi$  is, near  $b$ ,

$$\delta(y) = \frac{1}{2\sqrt{y-b}}(\tilde{\rho}(\sqrt{y-b}) + \tilde{\rho}(-\sqrt{y-b})) = \frac{\hat{\rho}(y-b)}{\sqrt{y-b}}$$

where  $\hat{\rho}$  is analytic near 0. Therefore, near  $b$ ,

$$\delta(x) = \frac{U}{\sqrt{b-x}} + U'\sqrt{b-x} + \dots$$

where  $U = \rho(c)/\sqrt{A}$ , and  $U'$  is linear in  $\rho(c), \rho'(c), \rho''(c)$  with coefficients depending on the derivatives of  $f$  at  $c$ . Near  $a$  we find

$$\delta(x) = U|f'(b)|^{-1/2} \frac{1}{\sqrt{x-a}} + (U'|f'(b)|^{-3/2} - \frac{3}{4}Uf''(b)|f'(b)|^{-5/2})\sqrt{x-a}$$

Writing  $s_n = -\text{sgn} \prod_{k=0}^{n-1} f'(f^k b)$ ,  $t_n = |\prod_{k=0}^{n-1} f'(f^k b)|^{-1/2}$ , we claim that near  $f^n b$  we have a singularity given for  $s_n(x - f^n b) < 0$  by 0, and for  $s_n(x - f^n b) > 0$  by

$$\delta(x) = \frac{Ut_n}{\sqrt{s_n(x - f^n b)}} + (U't_n^3 - \frac{3}{4}Ut_n \sum_{k=0}^{n-1} s_{k+1} \frac{f''(f^k b)}{|f'(f^k b)|} \frac{t_n^2}{t_k^2})\sqrt{s_n(x - f^n b)}$$

[to prove this we use induction on  $n$ , and the fact that, when  $f : x \mapsto y$  for  $x$  close to  $f^n b$  we have:

$$s_n(x - f^n b) = \frac{s_{n+1}(y - f^{n+1}b)}{|f'(f^n b)|} [1 - \frac{f''(f^n b)}{2|f'(f^n b)|^2}(y - f^{n+1}b)]$$

$$dx = \frac{dy}{|f'(f^n b)|} [1 - \frac{f''(f^n b)}{|f'(f^n b)|^2}(y - f^{n+1}b)] \quad ] .$$

Define now

$$\psi_n^{(0)}(x) = (1 - (\frac{x - f^n b}{w_n - f^n b})^2) \frac{\theta_n(x)}{\sqrt{s_n(x - f^n b)}}$$

$$\psi_n^{(1)}(x) = (1 - (\frac{x - f^n b}{w_n - f^n b})^2) \theta_n(x) \sqrt{s_n(x - f^n b)}$$

for  $s_n(x - f^n b) > 0$ , 0 otherwise. Then, the expected singularity of  $\delta$  near  $f^n b$  is given by

$$Ut_n\psi_n^{(0)} + (U't_n^3 - \frac{3}{4}Ut_n \sum_{k=0}^{n-1} s_{k+1} \frac{f''(f^k b)}{|f'(f^k b)|} \frac{t_n^2}{t_k^2})\psi_n^{(1)} = C_n^{(0)}\psi_n^{(0)} + C_n^{(1)}\psi_n^{(1)}$$

where  $C_0^{(0)} = U$ ,  $C_0^{(1)} = U'$ , and

$$\begin{aligned} C_{n+1}^{(0)} &= |f'(f^n b)|^{-1/2} C_n^{(0)} \\ C_{n+1}^{(1)} &= |f'(f^n b)|^{-3/2} C_n^{(1)} - \frac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) C_n^{(0)} \\ &= |f'(f^n b)|^{-3/2} (C_n^{(1)} - \frac{3}{4} s_{n+1} \frac{f''(f^n b)}{|f'(f^n b)|} C_n^{(0)}) \end{aligned}$$

Let

$$f(\psi_n^{(0)}(x) dx) = \tilde{\psi}_{n+1}^{(0)}(x) dx \quad , \quad f(\psi_n^{(1)}(x) dx) = \tilde{\psi}_{n+1}^{(1)}(x) dx$$

and write

$$\begin{aligned} \tilde{\psi}_{n+1}^{(0)} &= |f'(f^n b)|^{-1/2} \psi_{n+1}^{(0)} - \frac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) \psi_{n+1}^{(1)} + \chi_n^{(0)} \\ \tilde{\psi}_{n+1}^{(1)} &= |f'(f^n b)|^{-3/2} \psi_{n+1}^{(1)} + \chi_n^{(1)} \end{aligned}$$

The density of  $f(C_n^{(0)} \psi_n^{(0)}(x) dx + C_n^{(1)} \psi_n^{(1)}(x) dx)$  is then

$$C_{n+1}^{(0)} \psi_{n+1}^{(0)} + C_{n+1}^{(1)} \psi_{n+1}^{(1)} + C_n^{(0)} \chi_n^{(0)} + C_n^{(1)} \chi_n^{(1)}$$

The functions  $\chi_n^{(0)}, \chi_n^{(1)}$  have been constructed such that they and their first derivatives  $\chi_n^{(0)'}, \chi_n^{(1)'}$  have the properties of Lemma 11. Namely,  $\chi_n^{(0)}, \chi_n^{(1)}, \chi_n^{(0)'}, \chi_n^{(1)'}$  are continuous with bounded variation on  $[a, b]$  uniformly in  $n$ , they vanish at  $a, b$ , and if  $n \geq 1$  they extend to holomorphic functions on the appropriate  $D_\alpha$ , with uniform bounds.

Let  $\mathcal{A}'_1 \subset \mathcal{A}_1$  consist of the  $(\phi_\alpha)$  such that the derivatives  $\phi'_{-1}, \phi'_{-2}$  of  $\phi_{-1}, \phi_{-2}$  vanish at  $\pi_b^{-1}b$  and  $\pi_a^{-1}a$  respectively. Let also  $\mathcal{A}'_2$  consist of the sequences  $(c_n^{(0)}, c_n^{(1)})$ , with  $c_n^{(0)}, c_n^{(1)} \in \mathbf{C}$ ,  $n = 0, 1, \dots$  such that

$$\|(c_n^{(0)}, c_n^{(1)})\|'_2 = \sup_{n \geq 0} \delta^n (|c_n^{(0)}| + |c_n^{(1)}|) < \infty$$

If  $\Phi' = ((\phi_\alpha), (c_n^{(0)}, c_n^{(1)})) \in \mathcal{A}' = \mathcal{A}'_1 \oplus \mathcal{A}'_2$  we let  $\|\Phi'\|' = \|(\phi_\alpha)\|_1 + \|(c_n^{(0)}, c_n^{(1)})\|'_2$ , making  $\mathcal{A}'$  into a Banach space. We may now proceed as in Section 12, replacing  $\mathcal{A}$  by  $\mathcal{A}'$ , and defining  $\mathcal{L}' : \mathcal{A}' \mapsto \mathcal{A}'$  in a way similar to  $\mathcal{L} : \mathcal{A} \mapsto \mathcal{A}$ , but with (ii), (v), (vi) replaced as follows:

$$(ii) \phi_0 \Rightarrow \left( (\hat{c}_0^{(0)}, \hat{c}_0^{(1)}) = (U, U'), \hat{\phi}_{-1} = \pm \frac{\phi_0}{|f'|} \circ \tilde{f}_{-1}^{-1} - U(\pm \frac{1}{2} \psi_0^{(0)} \circ \pi_b) - U'(\pm \frac{1}{2} \psi_0^{(1)} \circ \pi_b) \right)$$

so that  $\hat{\phi}_{-1}$  is holomorphic in  $\pi_b^{-1}D_{-1}$  with vanishing derivative at  $\pi_b^{-1}b$

$$\begin{aligned} (v) (c_0^{(0)}, c_0^{(1)}) &\Rightarrow \left( (\hat{c}_1^{(0)}, \hat{c}_1^{(1)}) = (|f'(b)|^{-1/2} c_0^{(0)}, |f'(b)|^{-3/2} c_0^{(1)} - \frac{3}{4} |f'(b)|^{-5/2} f''(b) c_0^{(0)}), \right. \\ \chi_0 &= \pm \frac{1}{2} c_0^{(0)} \left( \frac{\psi_0^{(0)}}{|f'|} \circ \pi_b \circ \tilde{f}_{-2}^{-1} - |f'(b)|^{-1/2} \psi_1^{(0)} \circ \pi_a + \frac{3}{4} |f'(b)|^{-5/2} f''(b) \psi_1^{(1)} \circ \pi_a \right) \\ &\quad \left. \pm \frac{1}{2} c_0^{(1)} \left( \frac{\psi_0^{(1)}}{|f'|} \circ \pi_b \circ \tilde{f}_{-2}^{-1} - |f'(b)|^{-3/2} \psi_1^{(1)} \circ \pi_a \right) \text{ in } \pi_a^{-1}D_{-2} \right) \end{aligned}$$

$$\begin{aligned}
& \text{(vi)} \quad (c_n^{(0)}, c_n^{(1)}) \Rightarrow \\
& \left( (\hat{c}_{n+1}^{(0)}, \hat{c}_{n+1}^{(1)}) = (|f'(f^n b)|^{-1/2} c_n^{(0)}, |f'(f^n b)|^{-3/2} c_n^{(1)} - \frac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) c_n^{(0)}), \right. \\
& \chi_{n\alpha} = c_n^{(0)} \left[ \frac{\psi_n^{(0)}}{|f'|} \circ f_n^{-1} - |f'(f^n b)|^{-1/2} \psi_{n+1}^{(0)} + \frac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) \psi_{n+1}^{(1)} \right] \\
& \left. + c_n^{(1)} \left[ \frac{\psi_n^{(1)}}{|f'|} \circ f_n^{-1} - |f'(f^n b)|^{-3/2} \psi_{n+1}^{(1)} \right] \quad \text{in } D_\alpha \text{ if } V_\alpha \subset \{x : \theta_n(f_n^{-1}x) > 0\}, 0 \text{ otherwise} \right) \\
& \text{if } n \geq 1.
\end{aligned}$$

We write then

$$\mathcal{L}'\Phi' = \tilde{\Phi}' = ((\tilde{\phi}_\alpha), (\tilde{c}_n^{(0)}, \tilde{c}_n^{(1)}))$$

where

$$\begin{aligned}
\tilde{\phi}_{-2} &= \hat{\phi}_{-2} + \chi_0, & \tilde{\phi}_{-1} &= \hat{\phi}_{-1} \\
\tilde{\phi}_\alpha &= \sum_{\beta: fV_\beta = V_\alpha} \hat{\phi}_{\beta\alpha} + \hat{\phi}_\alpha + \sum_{n \geq 1} \chi_{n\alpha} & \text{if order } \alpha \geq 0 \\
(\tilde{c}_n^{(0)}, \tilde{c}_n^{(1)}) &= (\hat{c}_n^{(0)}, \hat{c}_n^{(1)}) & \text{for } n \geq 0
\end{aligned}$$

With the above definitions and assumptions we find, by analogy with Theorem 13, that  $\mathcal{L}' : \mathcal{A}' \rightarrow \mathcal{A}'$  has essential spectral radius  $\leq \max(\gamma^{-1}, \delta\alpha^{1/2})$ . There is (see Proposition 15) a simple eigenvalue 1, and the rest of the spectrum has radius  $< 1$ . It is convenient to denote by  $\Phi^0 = ((\phi_\alpha^0), (c_n^{0(0)}, c_n^{0(1)}))$  the eigenfunction to the eigenvalue 1. We find again that  $\phi^0 = \Delta(\phi_\alpha^0)$  is continuous, of bounded variation, and satisfies  $\phi^0(a) = \phi^0(b) = 0$ , but we can say more. Using the notation in the proof of Proposition 15, we have again

$$\gamma_j^0 = \sum_k \mathcal{L}_{jk} \gamma_k^0 + \eta_j$$

with  $\eta_j = \sum_{n=0}^\infty \eta_{jn}$ , but now  $\eta_{jn} = c_n^{0(0)} \chi_n^{(0)} + c_n^{0(1)} \chi_n^{(1)} |W_j$  for  $n \geq 1$ , so that the  $\eta_j$  have derivatives  $\eta_j' \in \mathcal{H}_j$ . The derivatives  $\gamma_j^{0'}$  of the  $\gamma_j^0$  are measures satisfying

$$\gamma_j^{0'} = \sum_k \mathcal{L}'_{jk} \gamma_k^{0'} + \eta_j^*$$

The operator  $\mathcal{L}'_{jk}$  has the same form as  $\mathcal{L}_{jk}$ , but with an extra denominator  $f' \circ (f^{-1})_{kj}$ , and therefore  $\mathcal{L}'_* = (\mathcal{L}'_{jk})$  acting on measures has spectral radius  $\leq \alpha < 1$ . The term  $\eta_j^*$  is the sum of  $\eta_j'$  and a term  $\sum_k \mathcal{L}_{kj}^* \gamma_k^0$  where  $\mathcal{L}_{kj}^*$  involves the derivative of  $|f' \circ (f^{-1})_{kj}|^{-1}$  so that  $\eta_j^* \in \mathcal{H}_j$ . The operator  $\mathcal{L}'_*$  also maps  $\mathcal{H}$  to  $\mathcal{H}$  and, by the same argument as for  $\mathcal{L}_*$ , has essential spectral radius  $< 1$  on  $\mathcal{H}$ . Furthermore, 1 cannot be an eigenvalue since  $\mathcal{L}'_*$  has spectral radius  $< 1$  on measures. It follows that  $(\gamma^{0'}) = (\gamma_j^{0'}) = (1 - \mathcal{L}'_*)^{-1}(\eta_j^*) \in \mathcal{H}$ . Therefore, the derivative  $\phi^{0'}$  of  $\phi^0$  may have discontinuities only on the orbit of  $u_1$ , and hyperbolicity again shows that this cannot happen. In conclusion,  $\phi^0$  and its derivative  $\phi^{0'}$  are both of bounded variation, continuous, and vanishing at  $a, b$ .

A discussion similar to the above shows that the equation  $\gamma = (1 - \mathcal{L}'_*)^{-1} \eta^*$  also defines  $\gamma$  with finite norm in  $\mathcal{A}_1$ , and this  $\gamma$  must coincide with  $(\gamma^{0'})$  as a measure. Therefore the family of derivatives  $(\phi_\alpha^{0'})$  is an element of  $\mathcal{A}_1$ . [For simplicity, we have written  $\phi_{-1}^{0'}$ ,  $\phi_{-2}^{0'}$  for the functions which, under application of  $\Delta$ , give the derivative of  $\Delta\phi_{-1}^0$ ,  $\Delta\phi_{-2}^0$ ].  $\square$

## B Appendix (proof of Remark 16(b)).

If  $u \in \tilde{H}$  and  $\psi_{(u\pm)}$  is defined as in Remark 16(b), we want to show that there is a unique  $(\phi_\alpha)$  in  $\mathcal{A}_1$  such that  $\phi_\alpha = \psi_{(u\pm)}|V_\alpha$  for all  $\alpha$ . Furthermore  $\|(\phi_\alpha)\|_1$  is bounded uniformly for  $u \in \tilde{H}$ , provided we assume  $1 < \gamma < \min(\beta^{-1}, \alpha^{-1/2})$ .

Note that uniqueness is automatic, and that  $\phi_\alpha = 0$  unless  $\text{order } V_\alpha > 0$ . Omitting the  $\pm$  we let

$$f(\psi_{(f^n u)}(x) dx) = [|f'(f^n u)|^{-1/2} \psi_{(f^{n+1} u)}(x) + \chi_{(f^n u)}(x)] dx$$

For  $n \geq 0$  there is a unique  $\omega_{un}$  such that  $f^{n+1}(\omega_{un} dx) = \prod_{k=0}^{n-1} |f'(f^k u)|^{-1/2} \chi_{(f^n u)} dx$  and  $[f^k u - c] \times [\text{supp } f^k(\omega_{un}(x) dx) - c] > 0$  for  $0 \leq k \leq n$ . Furthermore  $\psi_{(u)} = \sum_{n=0}^{\infty} \omega_{un}$  where the sum restricted to each  $V_\alpha$  is finite. If  $[\chi_{(f^n u)}]$  denotes the element of  $\mathcal{A}_1$  corresponding to  $\chi_{(f^n u)}$ , we find that  $\|[\chi_{(f^n u)}]\|_1$  is bounded uniformly in  $n$  and  $u$ . Also note that we obtain  $\omega_{un}$  from  $\prod_{k=0}^{n-1} |f'(f^k u)|^{-1/2} \chi_{(f^n u)}$  by multiplying with  $\prod_{k=0}^{n-1} |f'(f^k u)|$  (up to a factor bounded uniformly in  $n$  because of hyperbolicity) and composing with  $f^{n+1}$  (restricted to a small interval  $J$  such that  $f^{n+1}|_J$  is invertible). We have thus

$$\|[\omega_{un}]\|_1 \leq \text{const } \gamma^n \prod_{k=0}^{n-1} |f'(f^k u)|^{-1/2}$$

where  $[\omega_{un}]$  is the element of  $\mathcal{A}_1$  corresponding to  $\omega_{un}$  [This is because the replacement of  $|V_\alpha|$  by  $|(f|J)^{-n-1} V_\alpha|$  in the definition of  $\|\cdot\|_1$  is compensated up to a multiplicative constant by the factor  $\prod_{k=0}^{n-1} |f'(f^k u)|$ ]. Thus

$$\|[\omega_{un}]\|_1 \leq \text{const } (\gamma \alpha^{1/2})^n$$

Since  $\gamma < \alpha^{-1/2}$  we find that  $\sum_n \|[\omega_{un}]\|_1 < \text{constant}$  independent of  $u$ . Therefore, since  $(\phi_\alpha) = \sum_n [\omega_{un}]$ , we see that  $\|(\phi_\alpha)\|_1$  is bounded independently of  $u$ .  $\square$



## C Appendix (proof of Remark 16(c)).

We consider a one-parameter family  $(f_\kappa)$  of maps, reducing to  $f = f_0$  for  $\kappa = 0$ . We assume that  $(\kappa, x) \mapsto f_\kappa x$  is real-analytic. For  $\kappa$  close to 0,  $f_\kappa$  has a critical point  $c_\kappa$  and maps  $[a_\kappa, b_\kappa]$  to itself, with  $b_\kappa = f_\kappa c_\kappa, a_\kappa = f_\kappa^2 c_\kappa$ . There is (by hyperbolicity of  $H$  with respect to  $f$ ) a homeomorphism  $\xi_\kappa : H \rightarrow H_\kappa$  where  $H_\kappa$  is an  $f_\kappa$ -invariant hyperbolic set for  $f_\kappa$  and  $f_\kappa \circ \xi_\kappa = \xi_\kappa \circ f$  on  $H$ . We shall consider a compact set  $K$  of values of  $\kappa$  such that  $f_\kappa a_\kappa \in \tilde{H}_\kappa$ ; we let  $K \ni 0$ ,  $K$  of small diameter, and assume now  $\kappa \in K$ . We may in a natural way define a Banach space  $\mathcal{A}_\kappa = \mathcal{A}_{\kappa 1} \oplus \mathcal{A}_2$  and an operator  $\mathcal{L}_\kappa : \mathcal{A}_\kappa \rightarrow \mathcal{A}_\kappa$  associated with  $f_\kappa$  so that  $\mathcal{A}_\kappa, \mathcal{L}_\kappa$  reduce to  $\mathcal{A}, \mathcal{L}$  for  $\kappa = 0$ . Note that, since  $\kappa \in K$  is close to 0, we may assume that the constants  $A, \alpha$  in the definition (Section 4) of hyperbolicity, and the constants  $B, \beta$  (Section 7) are uniform in  $\kappa$ .

Let  $\eta_{\kappa, -2}$  be a biholomorphic map of the complex neighborhood  $D_{-2}$  of  $[a, u_1]$  to the complex neighborhood  $D_{\kappa, -2}$  of the corresponding interval  $[a_\kappa, u_{\kappa 1}]$ , and lift  $\eta_{\kappa, -2}$  to a holomorphic map  $\tilde{\eta}_{\kappa, -2} : \pi_a^{-1} D_{-2} \rightarrow \pi_{a_\kappa}^{-1} D_{\kappa, -2}$ . We also lift  $\eta_{\kappa, -1} = f_\kappa^{-1} \circ \eta_{\kappa, -2} \circ f$  to

$$\tilde{\eta}_{\kappa, -1} = \tilde{f}_{\kappa, -2}^{-1} \circ \eta_{\kappa, -2} \circ \tilde{f}$$

where the notation is that of Section 12, with obvious modification. We write

$$\tilde{\eta}_{\kappa 0} = \tilde{f}_{\kappa, -1}^{-1} \circ \tilde{\eta}_{\kappa, -1} \circ \tilde{f}_{-1}$$

and

$$\tilde{\eta}_{\kappa \beta} = (f_\kappa|_{V_{\kappa \beta}})^{-1} \circ \tilde{\eta}_{\kappa \alpha} \circ f|_{V_\beta}$$

if order  $\beta > 0$  and  $fV_\beta = V_\alpha$ . We have defined  $\eta_{\kappa \alpha}$  above for  $\alpha = -1, -2$ , and we let  $\eta_{\kappa \alpha} = \tilde{\eta}_{\kappa \alpha}$  when order  $\alpha \geq 0$ .

We introduce a map  $\eta_\kappa : \mathcal{A}_{\kappa 1} \rightarrow \mathcal{A}_1$  by

$$\eta_\kappa(\phi_{\kappa \alpha}) = ((\phi_{\kappa \alpha} \circ \tilde{\eta}_{\kappa \alpha}) \cdot \eta'_{\kappa \alpha})$$

so that  $\mathcal{L}_\kappa^\times = (\eta_\kappa, \mathbf{1})\mathcal{L}_\kappa(\eta_\kappa^{-1}, \mathbf{1})$  acts on  $\mathcal{A}$ . Using the decomposition

$$\mathcal{L}_\kappa = \begin{pmatrix} \mathcal{L}_{\kappa 0} + \mathcal{L}_{\kappa 1} & \mathcal{L}_{\kappa 2} \\ \mathcal{L}_{\kappa 3} & \mathcal{L}_{\kappa 4} \end{pmatrix}$$

as in Section 12, we define  $L_\kappa^\times$  on  $\mathcal{A}_1$  by

$$\begin{aligned} L_\kappa^\times(\phi_\alpha) &= \eta_\kappa(\mathcal{L}_{\kappa 0} + \mathcal{L}_{\kappa 1})\eta_\kappa^{-1}(\phi_\alpha) + (\eta_\kappa^{-1}\phi_\alpha)_0(c_\kappa) \cdot \eta_\kappa \mathcal{L}_{\kappa 2} \left( \left| \frac{1}{2} f_\kappa''(c_\kappa) \prod_{k=0}^{n-1} f'_\kappa(f_\kappa^k b_k) \right|^{-1/2} \right) \\ &= \mathcal{L}_0(\phi_\alpha) + \eta_\kappa \mathcal{L}_{\kappa 1} \eta_\kappa^{-1}(\phi_\kappa) + \eta'_{\kappa 0}(c_\kappa)^{-1} \phi_0(c_\kappa) \cdot \eta_\kappa \mathcal{L}_{\kappa 2} \left( \left| \frac{1}{2} f_\kappa''(c_\kappa) \prod_{k=0}^{n-1} f'_\kappa(f_\kappa^k b_k) \right|^{-1/2} \right) \end{aligned}$$

$L_\kappa^\times$  is a compact perturbation of  $\mathcal{L}_{\kappa 0}$ , and has therefore essential spectral radius  $\leq \gamma^{-1}$ . If  $(\phi_\alpha)$  is a (generalized) eigenfunction of  $L_\kappa^\times$  to the eigenvalue  $\mu$ , then

$$((\phi_\alpha), \eta_{\kappa 0}(c_\kappa)^{-1} \phi_0(c_\kappa) \cdot (|\frac{1}{2} f_\kappa''(c_\kappa) \prod_{k=0}^{n-1} f_\kappa'(f_\kappa^k b_k)|^{-1/2}))$$

is a (generalized) eigenfunction of  $\mathcal{L}_\kappa^\times$  to the same eigenvalue  $\mu$ . We have thus a multiplicity-preserving bijection of the eigenvalues  $\mu$  of  $L_\kappa^\times$  and  $\mathcal{L}_\kappa^\times$  when  $|\mu| > \max(\gamma^{-1}, \delta\alpha^{1/2})$ . In particular, 1 is a simple eigenvalue of  $L_\kappa^\times$  for the values of  $\kappa$  considered (a compact neighborhood  $K$  of 0).

The operator  $L_\kappa^\times$  acting on  $\mathcal{A}_1$  depends continuously on  $\kappa$ . [This is because  $\hat{\phi}_{\kappa\alpha}$ ,  $\chi_{\kappa 0}$ ,  $\chi_{\kappa n\alpha}$  depend continuously on  $\kappa$  (in particular, the  $\chi_{\kappa n\alpha}$  for large  $n$  are uniformly small). Note however that  $\mathcal{L}_\kappa^\times$  does not depend continuously on  $\kappa$  because the continuity of  $f_\kappa'(f_\kappa^k b_\kappa)$  is not uniform in  $k$ ]. There is  $\epsilon > 0$  such that  $L_\kappa^\times$  has no eigenvalue  $\mu_\kappa$  with  $|\mu_\kappa - 1| < \epsilon$  except the simple eigenvalue 1 [otherwise the continuity of  $\kappa \rightarrow L_\kappa^\times$  would imply that 1 has multiplicity  $> 1$  for some  $\kappa$ ]. Therefore, the 1-dimensional projection corresponding to the eigenvalue 1 of  $L_\kappa^\times$  depends continuously on  $\kappa$ , and so does the eigenvector  $\Phi_\kappa^\times = (\eta_\kappa, 1)\Phi_\kappa^0$  of  $\mathcal{L}_\kappa^\times$ , where  $\Phi_\kappa^0$  denotes the eigenvector the the eigenvalue 1 of  $\mathcal{L}_\kappa$  normalized so that  $w_\kappa \Phi_\kappa^0 \geq 0$  and  $\int w_\kappa \Phi_\kappa^0 = 1$ , with the obvious definition of  $w_\kappa$  (involving the spikes  $\psi_{\kappa n}$  associated with  $f_\kappa$ ).

Note that a number of results have been obtained earlier on the continuous dependence of the a.c.i.m.  $\rho$  on parameters. I am indebted to Viviane Baladi for communicating the references [25], [27], [15], and also [26].

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